Rudin Exercise Reviews

Richard K. Yu

December 11, 2025

Chapter 1: The Real and Complex Number Systems

1. If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.
   Proof. Suppose, for contradiction, that $r + x$ is rational. Then since the rationals are closed, we see that $(r + x) - r = x$ is rational, a contradiction. A similar argument holds for rx.

   $\square$

2. Prove that there is no rational number whose square is 12.
   Proof. Suppose, for contradiction, that there is a rational number whose square is 12. Then, we can write $\left(\frac{p}{q}\right)^2 = 12$ for integers p, q with nonzero q. However, this means that

   $$p^2 = q^2 \cdot 2^2 \cdot 3$$

   If we consider the unique prime factorization of p^2, we see that the left hand side has an even power of three while the right hand side has an odd power of three. This is impossible because the factorization is unique.

   $\square$

3. Prove Proposition 1.15.
   Proof.

   (a) If $x \neq 0$ and $xy = xz$, then $y = z$.
   Observe that

   $$y = \left(\frac{1}{x} \cdot x\right) y = \frac{1}{x}(xy) = \frac{1}{x}(xz) = \left(\frac{1}{x} \cdot x\right) z = z$$

   (b) If $x \neq 0$ and $xy = x$, then $y = 1$.
   Simply take $z = 1$ and apply (a).

   (c) If $x \neq 0$ and $xy = 1$, then $y = \frac{1}{x}$.
   By the field axiom M5, there exists $\frac{1}{x} \in F$ such that $x \cdot \frac{1}{x} = 1$. So $xy = x \cdot \frac{1}{x}$ and $y = \frac{1}{x}$ by (a).

   (d) If $x \neq 0$, then $1/(1/x) = x$.
   We know that $\frac{1}{x} \cdot (1/(1/x)) = 1 = \frac{1}{x} \cdot x$. Hence $(1/(1/x)) = x$ by (a).

$\square$

4. Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.
   Proof. Since E is nonempty, we may choose $x \in E$. Then, by the definitions of upper and lower bound we have $\alpha \leq x \leq \beta$. So $\alpha \leq \beta$ by transitivity.

   $\square$

5. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that $\inf A = -\sup(-A)$.
   Proof. Since A is nonempty set of real numbers that is bounded below, $\inf A$ exists by the LUB property. By definition, we have $\inf A \leq x$ for all $x \in A$. Hence, $-\inf A \geq -x$, for all $x \in A$. Thus, $-\inf A$ is an upper bound for $-A$.
   Suppose that U is an upper bound of $-A$. This means that $U \geq -x$ for all $x \in A$. As a result, $-U \leq x$ for all $x \in A$ and $-U$ is a lower bound of A. By definition, we see that $-U \leq \inf A$, and this implies $-\inf A \leq U$. Therefore, $-\inf A$ is the least upper bound of $-A$. Hence.
   $$-\inf A = \sup(-A)$$
   as desired.

   $\square$

6. Fix $b > 1$.

   (a) If m, n, p, q are integers, $n > 0, q > 0$, and $r = \frac{m}{n} = \frac{p}{q}$, prove that
   $$(b^m)^{\frac{1}{n}} = (b^p)^{\frac{1}{q}}$$
   Hence it makes sense to define $b^r = (b^m)^{\frac{1}{n}}$.
   Lemma. We have $b^{x \cdot \frac{1}{y}} = b^{\frac{1}{y} \cdot x}$ for integers x, y and y nonzero.
   We consider the case where x, y are positive, the remaining cases are analogous. In this case
   $$(b^{\frac{1}{y}})^x = \underbrace{b^{\frac{1}{y}} \cdots b^{\frac{1}{y}}}_{x \text{ times}} = (b^x)^{\frac{1}{y}}$$
   where the last equality follows from the Corollary to Theorem 1.21.
   Proof. Clearly, $mq = pn$. Hence, $b^{mq} = b^{pn}$. Taking the n^{th} root and then q^{th} root, we have $b^{(mq)\frac{1}{n}} = b^p$ and $(b^m)^{\frac{1}{n}} = (b^p)^{\frac{1}{q}}$. We can exchange roots and exponents since $b > 1$.

   $\square$

   (b) Prove that $b^{r+s} = b^r \cdot b^s$ if r and s are rational.
   Proof. Let $r = \frac{m}{n}$ and $s = \frac{p}{q}$ for integers m, n, p, q where n, q are nonzero. Then
   $$b^{r+s} = (b^{mq+pn})^{\frac{1}{nq}} = (b^{mq} b^{pn})^{\frac{1}{nq}} = b^r b^s$$

   $\square$

(c) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that

$$b^r = \sup B(r)$$

when r is rational. Hence it makes sense to define

$$b^x = \sup B(x)$$

for every real x.

Proof. Let r and t both be rational and $t \leq r$. Since $b > 1$ and $r - t \geq 0$, it follows that $b^{r-t} \geq 1$. Hence $b^r \geq b^t$ via part (b). This shows that $b^r = \max B(r)$, so $b^r = \sup B(r)$ as claimed.

$\square$

One point to note is that if $b > 1$ and $d > 0$ for rational d, then $b^d > 1$.

(d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

Proof. We want to show that $\sup B(x+y) = \sup B(x) \cdot \sup B(y)$. Let $b^s \in B(x)$ and $b^t \in B(y)$. Then, s and t are rational and satisfy $s \leq x$ and $t \leq y$. From part (c), we have $b^s \leq b^x$ and $b^t \leq b^y$. Using part (b), we have

$$b^s \cdot b^t = b^{s+t} \leq b^{x+y}$$

Hence, $\sup B(x+y)$ is an upper bound of $B(x) \cdot B(y)$. We have

$$\sup B(x+y) \geq \sup(B(x) \cdot B(y)) = \sup B(x) \cdot \sup B(y)$$

via Problem 8. Now, choose $r \in \mathbb{Q}$ such that $r < x + y$ and $\varepsilon > 0$ such that $r < x + y - \varepsilon$. By Theorem 1.20(b), we may choose $q_1, q_2 \in \mathbb{Q}$ such that $x - \frac{\varepsilon}{2} < q_1 \leq x$, $y - \frac{\varepsilon}{2} < q_2 \leq y$. Adding these inequalities, we have

$$r < x + y - \varepsilon < q_1 + q_2 \leq x + y$$

Thus

$$b^r < b^{q_1+q_2} = b^{q_1} \cdot b^{q_2} \in B(x) \cdot B(y)$$

Hence $\sup B(x+y) \leq \sup B(x) \cdot \sup B(y)$ and equality follows.

$\square$

7. Fix $b > 1$, $y > 0$ and prove that there is a unique real x such that $b^x = y$, by completing the following outline. (This x is called the *logarithm of y to the base b*.)

(a) For any positive integer n, $b^n - 1 \geq n(b-1)$.
Observe that

$$b^n - 1 = (b-1)\underbrace{(b^{n-1} + b^n + \ldots + b + 1)}_{n \text{ terms}} \geq n(b-1)$$

where the last inequality follows since $b > 1$.

(b) Hence $b - 1 \geq n(b^{\frac{1}{n}} - 1)$. This follows since $b^{\frac{1}{n}} > 1$.

(c) If $t > 1$ and $n > \frac{b-1}{t-1}$, then $b^{\frac{1}{n}} < t$.
From part (b), we have

$$n > \frac{b-1}{t-1} \geq \frac{n(b^{\frac{1}{n}}-1)}{t-1}$$

Dividing by n and multiplying through by $t-1$, we have $b^{\frac{1}{n}} < t$.

(d) If w is such that $b^w < y$, then $b^{w+\frac{1}{n}} < y$ for sufficiently large n; to see this, apply part (c) with $t = y \cdot b^{-w}$.
Multiplying on both sides by b^{-w}, we see that $t = y \cdot b^{-w} > 1$. By the Archimedean property, we can choose $n > \frac{b-1}{t-1}$. Applying part (c) yields

$$b^{\frac{1}{n}} < y \cdot b^{-w}$$

from which the desired conclusion follows.

(e) If $b^w > y$, then $b^{w-\frac{1}{n}} > y$ for sufficiently large n.
Set $t = y^{-1}b^w$. Then applying part (c) for an appropriate n, we have

$$b^{\frac{1}{n}} < y^{-1}b^w$$

Multiplying by b^{-w} on both sides and taking the reciprocal show $b^{w-\frac{1}{n}} > y$.

(f) Let A be the set of all w such that $b^w < y$, and show that $x = \sup A$ satisfies $b^x = y$.
Suppose, for contradiction, that $x = \sup A$ and $b^x \neq y$. If $b^x < y$, then by part (d), it follows that $b^{x+\frac{1}{n}} < y$ for sufficiently large n. Hence $x + \frac{1}{n} \in A$ and x is not an upper bound of A. On the other hand, suppose that $b^x > y$. By part (e), it follows that $b^{x-\frac{1}{n}} > y$ for sufficiently large n. But then $x - \frac{1}{n}$ is an upper bound for A since

$$b^{x-\frac{1}{n}} > y > b^w$$

Thus, x is not the least upper bound ($\lightning$). Therefore, we must have $b^x = y$. Note that we must have $x - \frac{1}{n} > w$ since $b > 1$.

(g) Prove that this x is unique.
The supremum of a set must be unique. Therefore, x is unique.

$\square$

8. Prove that no order can be defined in the complex field that turns it into an ordered field.
Proof. Suppose that it is possible to define an order which turns the complex field into an ordered field. Then, by Proposition 1.18(d), we must have for all nonzero $x \in \mathbb{C}$, that $x^2 > 0$. However, $i \neq 0$ and $i^2 = -1 < 0$ ($\lightning$). Therefore, we cannot define an order that makes the complex field into an ordered field.

$\square$

9. Suppose $z = a + bi$, $w = c + di$. Define $z < w$ if $a < c$ and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. (This type of order relation is called a *dictionary order*, or *lexicographic order*, for obvious reasons.)

Does this ordered set have the least-upper-bound property?
Proof. It is clear that for any $z, w \in \mathbb{C}$, we must have $z < w$, $z = w$, or $z > w$ based on this definition, since each of the possible combinations of inequalities and equalities are considered by dictionary order. Suppose that $x, y, z \in \mathbb{C}$ and $x < y$ and $y < z$. Denote $x = x_1 + x_2 i$, $y = y_1 + y_2 i$ and $z = z_1 + z_2 i$. We proceed by cases.

- If $y_1 > x_1$ and $z_1 > y_1$, then by transitivity of the real numbers we have $z_1 > x_1$. Thus $z > x$.
- If $y_1 > x_1$ and $z_1 = y_1$ but $z_2 > y_2$, then $z_1 > x_1$ so $z > x$ again.
- If $y_1 = x_1$ and $z_1 > y_1$, then $z_1 > x_1$ and $z > x$.
- Finally, if $y_1 = x_1$ and $y_2 > x_2$ and $z_1 = y_1$ and $z_2 > y_2$, then $z_1 = x_1$ and $z_2 > x_2$. Hence $z > x$.

This shows transitivity in all cases, so dictionary order turns the set of complex numbers into an ordered set.
This ordered set does not have the least-upper-bound property. Consider the set

$$S = \{1 + bi : b \in \mathbb{R}\}$$

This set is bounded above since 2 is an upper bound. However, it does not have a least upper bound because the possible values of b are not bounded.

□

10. Suppose $z = a + bi$, $w = u + iv$ and

$$a = \left(\frac{|w| + u}{2}\right)^{\frac{1}{2}}, \quad b = \left(\frac{|w| - u}{2}\right)^{\frac{1}{2}}$$

Prove that $z^2 = w$ if $v \geq 0$ and that $(\bar{z})^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception!) has two complex square roots.
Proof. We compute directly that

$$\begin{aligned} z^2 = (a + bi)^2 = a^2 - b^2 + 2abi &= u + 2i\left(\frac{|w|^2 - u^2}{4}\right)^{\frac{1}{2}} \\ &= u + 2i\left(\frac{u^2 + v^2 - u^2}{4}\right)^{\frac{1}{2}} \\ &= u + iv = w \end{aligned}$$

where $\sqrt{v^2} = v$ since $v \geq 0$. Also, we have

$$(\bar{z})^2 = u - i\sqrt{v^2} = u + iv$$

since $v \leq 0$.
So, given a complex number w with $v \geq 0$, there are two square roots of the form $a + bi$ and $-a - bi$. Likewise, if $v \leq 0$, the square roots are of the form $a - bi$ and $-a + bi$. The exception is if $z = 0$, for which there is only one square root.

□

11. If z is a complex number, prove that there exists $r \geq 0$ and a complex number w with $|w| = 1$ such that $z = rw$. Are w and r always uniquely determined by z?
Proof. If $|z| \neq 0$, take $r = |z|$ and $w = \frac{z}{|z|}$. By direct computation, we see that $z = rw$. In the case where $r = 0$, w is not unique. Otherwise, if $r \neq 0$, then w and r are uniquely determined. First, r must be unique since because $r = r|w| = |rw| = |z|$. Then, the relation $z = rw$ shows that $w = \frac{z}{r}$.

□

12. If $z_1, \ldots, z_n$ are complex, prove that

$$|z_1 + z_2 + \ldots + z_n| \leq |z_1| + |z_2| + \ldots + |z_n|$$

Proof. We proceed by induction. By Theorem 1.33(e), we know that $|z+w| \leq |z|+|w|$ for complex z, w. Suppose, for some $k \in \mathbb{N}$ that

$$\left| \sum_{i=1}^{k} z_i \right| \leq \sum_{i=1}^{k} |z_i|$$

Then, it follows that

$$|(z_1 + \ldots + z_k) + z_{k+1}| \leq |z_1 + \ldots + z_k| + |z_{k+1}| \leq |z_1| + \ldots + |z_k| + |z_{k+1}|$$

where we have used 1.33(e) first and the inductive hypothesis afterward.

□

13. If x, y are complex, prove that

$$||x| - |y|| \leq |x - y|$$

Proof. By Theorem 1.33(e), we know that

$$|(x - y) + y| \leq |x - y| + |y| \implies |x| - |y| \leq |x - y|$$

And a similar argument shows $|y| - |x| \leq |x - y|$, so the desired inequality follows.

□

14. If z is a complex number such that $|z| = 1$, that is, such that $z\bar{z} = 1$, compute

$$|1 + z|^2 + |1 - z|^2$$

Observe that

$$\begin{aligned} |1 + z|^2 + |1 - z|^2 &= (1 + z)(\overline{1 + z}) + (1 - z)(\overline{1 - z}) \\ &= (1 + z)(1 + \overline{z}) + (1 - z)(1 - \overline{z}) \\ &= 1 + z + \overline{z} + z\overline{z} + 1 - z - \overline{z} + z\overline{z} = 4 \end{aligned}$$

□

15. Under what conditions does equality hold in the Schwarz inequality?
    Recall the statement of the Schwarz inequality: if $a_1, \ldots, a_n$ and $b_1, \ldots b_n$ are complex numbers, then

$$\left|\sum_{j=1}^{n} a_j \overline{b_j}\right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2$$

Let us consider when the equation is equal:

$$\left|\sum_{j=1}^{n} a_j \overline{b_j}\right|^2 = \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2$$

Expanding expressions, we obtain:

$$(a_1\overline{b_1} + \ldots + a_n\overline{b_n})(\overline{a_1}b_1 + \ldots + \overline{a_n}b_n) = (a_1\overline{a_1} + \ldots a_n\overline{a_n})(b_1\overline{b_1} + \ldots b_n\overline{b_n})$$
$$\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i\overline{b_i})(\overline{a_j}b_j) = \sum_{i=1}^{n}\sum_{j=1}^{n}(a_i\overline{a_i})(b_j\overline{b_j})$$

In order for this equality to hold, we must have

$$a_i\overline{a_j}\overline{b_i}b_j = a_i\overline{a_i}b_j\overline{b_j} \implies a_jb_i = a_ib_j$$

This equality holds when $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are vectors which differ only by a constant.

16. Suppose $k \geq 3$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$, $|\mathbf{x} - \mathbf{y}| = d > 0$ and $r > 0$. Prove:

    (a) If $2r > d$, there are infinitely many $\mathbf{z} \in \mathbb{R}^k$ such that

$$|\mathbf{z} - \mathbf{x}| = |\mathbf{z} - \mathbf{y}| = r$$

Proof. Without loss of generality, put the midpoint of $\mathbf{x}$ and $\mathbf{y}$ as the origin. This means that $\frac{\mathbf{x}+\mathbf{y}}{2} = \mathbf{0}$, or $x_i = -y_i$ for all $i = 1, 2, \ldots, k$. Without loss of generality, we may also align the axis of the first coordinate with the line formed by $\mathbf{x}$ and $\mathbf{y}$. Consider points $\mathbf{z} = (z_1, \ldots, z_n)$ which are at a distance r from $\mathbf{x}$ or $\mathbf{y}$. We must have

$$\sum_{i=1}^{k}(x_i - z_i)^2 = r^2, \quad \sum_{i=1}^{k}(y_i - z_i)^2 = r^2$$

Since $\mathbf{x}$ and $\mathbf{y}$ are aligned with the first coordinate, by subtracting these equations from each other, we see that they reduce to the following:

$$(x_1 - z_1)^2 - (y_1 - z_1)^2 = 0 \implies x_1z_1 = 0$$

by expanding the binomials and substituting $x_1 = -y_1$. However, since $|\mathbf{x}-\mathbf{y}| > 0$, we must have $x_1 > 0$. Hence $z_1 = 0$. Finally, we see that the points $\mathbf{z}$ are determined by the equation

$$\sum_{i=2}^{k} z_i^2 = r^2 - x_1^2$$

There are infinitely points that satisfy this equation, as it describes an $(k-2)$-sphere with positive radius (as $|x_1 - y_1| = d$, this means $|x_1| = \frac{d}{2} < r$, since the origin lies between x_1 and y_1).

(b) If $2r = d$, there is exactly one such $\mathbf{z}$.
Using the same setup as in part (a), we reach the conclusion that

$$\sum_{i=2}^{k} z_i^2 = 0$$

Hence, the origin is the only point that satisfies this equation.

□

(c) If $2r < d$, there is no such $\mathbf{z}$.
Using the same setup as in part (a), we see that $r^2 - x_1^2 < 0$. Hence, if a solution were to exist, then the left side of $\sum_{i=2}^{k} z_i^2 = r^2 - x_1^2$ is non-negative while the right side is negative, which leads to a contradiction. Therefore, there is no such $\mathbf{z}$.

□

How must these statements be modified if k is 2 or 1?
If $k = 2$, then the case where $2r > d$ has two points $\mathbf{z}$ which satisfy the equality. If $k = 1$, then the case where $2r > d$ has no such $\mathbf{z}$. The other cases remain the same.

17. Prove that
$$|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$
If $\mathbf{x} \in \mathbb{R}^k$ and $\mathbf{y} \in \mathbb{R}^k$. Interpret this geometrically, as a statement about parallelograms.
Proof. Observe that
$$\begin{aligned} 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2 &= |\mathbf{x}|^2 + 2\mathbf{x} \cdot \mathbf{y} + |\mathbf{y}|^2 + |\mathbf{x}|^2 - 2\mathbf{x} \cdot \mathbf{y} + |\mathbf{y}|^2 \\ &= (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) + (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \\ &= |\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 \end{aligned}$$
Geometrically, this means that adding the squared lengths of the diagonals of the parallelogram formed by $\mathbf{x}$ and $\mathbf{y}$ is equal to twice the sum of the squared lengths of each of the sides.

□

18. If $k \geq 2$ and $\mathbf{x} \in \mathbb{R}^k$, prove that there exists $\mathbf{y} \in \mathbb{R}^k$ such that $\mathbf{y} \neq \mathbf{0}$ but $\mathbf{x} \cdot \mathbf{y} = 0$. Is this also true if $k = 1$?
Proof. Let $\mathbf{x} = (x_1, \ldots, x_n)$. Arrange the entries of $\mathbf{x}$ so that the nonzero entries come before the zero entries, we can make this rearrangement without loss of generality because the dot product of the rearrangements and our original two vectors will be the same. We construct $\mathbf{y}$ in the following way. If x_i and x_{i+1} are both nonzero, then $y_i = -x_{i+1}$ and $y_{i+1} = -x_i$. If x_i and x_{i+1} are both zero, choose any nonzero values for y_i and y_{i+1}. If x_i is nonzero and x_{i+1} is zero or $i + 1 > k$, set y_i to zero and y_{i+1} to

any nonzero value (if $i+1>k$, do nothing).
This is not true if $k=1$. In this case we must have $x_1y_1=0$ and $y_1\neq 0$. By the field axioms, we must have $x_1=0$. Hence, for any nonzero value of x_1 such a y_1 will not exist.

□

19. Suppose $\mathbf{a}\in\mathbb{R}^k$, $\mathbf{b}\in\mathbb{R}^k$. Find $\mathbf{c}\in\mathbb{R}^k$ and $r>0$ such that

$$|\mathbf{x}-\mathbf{a}|=2|\mathbf{x}-\mathbf{b}|$$

if and only if $|\mathbf{x}-\mathbf{c}|=r$.
Solution. We consider the solution given by Rudin and choose $3\mathbf{c}=4\mathbf{b}-\mathbf{a}$ and $3r=2|\mathbf{b}-\mathbf{a}|$. Now we verify the claim. Suppose that $|\mathbf{x}-\mathbf{a}|=2|\mathbf{x}-\mathbf{b}|$. Observe that

$$3|\mathbf{x}-\mathbf{c}|=3r$$
$$|3\mathbf{x}-4\mathbf{b}+\mathbf{a}|=2|\mathbf{b}-\mathbf{a}|$$

Squaring the equation and expanding yields

$$9\mathbf{x}\cdot\mathbf{x}-24\mathbf{x}\cdot\mathbf{b}+6\mathbf{x}\cdot\mathbf{a}+16\mathbf{b}\cdot\mathbf{b}-8\mathbf{b}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{a}=4\mathbf{b}\cdot\mathbf{b}-8\mathbf{b}\cdot\mathbf{a}+4\mathbf{a}\cdot\mathbf{a}$$

Combining terms and rearranging, we see that

$$(-3\mathbf{x}\cdot\mathbf{x}+6\mathbf{x}\cdot\mathbf{a}-3\mathbf{a}\cdot\mathbf{a})+(12\mathbf{x}\cdot\mathbf{x}-24\mathbf{x}\cdot\mathbf{b}+12\mathbf{b}\cdot\mathbf{b})=0$$

Dividing through by 3, we see that

$$(\mathbf{x}-\mathbf{a})\cdot(\mathbf{x}-\mathbf{a})=4(\mathbf{x}-\mathbf{b})\cdot(\mathbf{x}-\mathbf{b})$$

which simplifies to the desired equality. To show the other direction, we may simply start from the end and work in reverse, since we are dealing with equalities along the way.

□

20. With reference to the Appendix, suppose that property (III) were omitted from the definition of a cut. Keep the same definitions of order and addition. Show that the resulting ordered set has the least-upper-bound property, that addition satisfies axioms (A1) to (A4) (with a slightly different zero-element!) but that (A5) fails.
Proof. Let A be a nonempty set of $\mathbb{R}$, where we are using this modified definition of a cut, which is bounded above by $\beta\in\mathbb{R}$. We define

$$\gamma:=\bigcup_{\alpha\in A}\alpha$$

To see that $\gamma\in\mathbb{R}$, it is nonempty since each $\alpha\in A$ is nonempty. Further, $\gamma\neq\mathbb{Q}$ otherwise $\beta\subset\gamma$. Next, suppose that $p\in\gamma$. Since $p\in\gamma$, there exists some $\alpha\in A$ such that $p\in\alpha$. Since $\alpha\in\mathbb{R}$, $q\in\mathbb{Q}$ and $q<p$ implies $q\in\alpha$. Since $\alpha\subseteq\gamma$, we have $q\in\gamma$. Now, we show that $\gamma=\sup A$. First, γ is an upper bound since we have $\alpha\subseteq\gamma$ for all $\alpha\in A$, by the definition of union. Suppose γ_2 is an upper bound for A. Taking any

$p \in \gamma$, there exists some $\alpha \in A$ such that $p \in \alpha$. Since γ_2 is an upper bound for A, we have $\alpha \subseteq \gamma_2$ and so $p \in \gamma_2$. Hence $\gamma \subseteq \gamma_2$. So, γ is the least upper bound.

Observe that (A1) through (A3) hold based on the same arguments made by Rudin in the appendix (these arguments do not rely on Property III). Now, if Property III does not hold, this means that $p \in \alpha$ and $p \geq r$ for all $r \in \alpha$ is possible. From this idea, we define 0^* to be the set of *nonpositive* rational numbers. However, observe that (A5), the axiom for the existence of additive inverses, fails because there is no additive inverse for the set of negative rational numbers. To see this, call the set of negative rationals α and suppose, for contradiction, that its inverse, β, exists. Then, we must have $0 \in \alpha + \beta$. Hence, there exist $a \in \alpha$ and $b \in \beta$ satisfying $a + b = 0$. Hence, $b > 0$. However, since α consists of all negative rationals, $\frac{a}{2} \in \alpha$ and $\frac{a}{2} + b > 0$, showing that a positive number exists in $\alpha + \beta$. But $\alpha + \beta = 0^*$, so this is a contradiction.

$\blacksquare$

Chapter 2: Basic Topology

1. Prove that the empty set is a subset of every set.
   Proof. Let S be any set. For all x, $x \in \varnothing$ is false. Thus, $x \in \varnothing \implies x \in S$ is true since the antecedent is false. Therefore, $\varnothing \subseteq S$.

2. A complex number z is said to be *algebraic* if there are integers $a_0, \ldots, a_n$, not all zero, such that

   $$a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n = 0$$

   Prove that the set of all algebraic numbers is countable. *Hint:* For every positive integer N there are only finitely many equations with

   $$n + |a_0| + |a_1| + \ldots + |a_n| = N$$

   Proof. Suppose that z is algebraic. Then there exists an $n \in \mathbb{N}$ for which z must be a root to the equation

   $$a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n = 0$$

   By the fundamental theorem of algebra, this equation has at most n roots, meaning it has a finite number of roots. Since $\mathbb{Z}$ is countable, Theorem 2.13 allows us to conclude that the set of $(n+1)$-tuples of integers $\mathbb{Z}^{n+1}$ is also countable. There is an obvious bijection between $\mathbb{Z}^{n+1}$ and equations that z satisfies, which shows that there are countably many such equations. Since each equation has finite solutions, the set of algebraic numbers satisfying polynomial equations with $n+1$ integer coefficients is the countable union of sets of finite sets, which is again countable by the Corollary to Theorem 2.12.
   Since this argument applies to each $n \in \mathbb{N}$, we see that the set of all algebraic numbers is the countable union of countable sets, which must again be countable.

   $\square$

3. Prove that there exists real numbers which are not algebraic.
   Proof. Suppose, for contradiction, that every real number is algebraic. From the previous proof, we know that the set of algebraic numbers is countable. Since the real numbers are an infinite set, they are a countably infinite set by Theorem 2.8. However, from the Corollary to Theorem 2.43, the set of real numbers is uncountable. ($\lightning$)

   $\square$

4. Is the set of all irrational real numbers countable?
   No. The set of all irrational real numbers is uncountable. Suppose, for contradiction, that irrationals are countable. Since $\mathbb{R}$ is the union of the rationals and irrationals, it is the finite union of countable sets, which means $\mathbb{R}$ is countable. ($\lightning$)

   $\square$

5. Construct a bounded set of real numbers with exactly three limit points.
   Consider the set

   $$\left\{ i + \frac{1}{n} \mid n \in \{2, 3, 4, \ldots\}, i \in \{0, 1, 2\} \right\}$$

6. Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\overline{E}$ have the same limit points. Do E and E' always have the same limit points?
Proof. Suppose that $x \in (E')^c$. Consequently, x is not a limit point of E. So, there exists a neighborhood $N_r(x)$ such that $N_r(x) \cap (E \setminus \{x\}) = \varnothing$. Thus, if $q \in N_r(x)$ and $q \neq x$, then $q \notin E$. This means that $q \in E^c$. However, since $N_r(x)$ is open, there exists a neighborhood $N_{r_q}(q) \subseteq N_r(x)$. Hence q is not a limit point of E, so $q \in (E')^c$. We conclude that $N_r(x) \subseteq (E')^c$. Since $(E')^c$ is open, by Theorem 2.23, E' is closed.
It is clear that the set of limit points of E is contained in the set of limit points for $\overline{E}$, since $E \subseteq \overline{E}$. Conversely, suppose that x is not a limit point of E. Then, there exists a neighborhood $N_r(x) \setminus \{x\} \subseteq E^c$. This neighborhood shows that x is not a limit point of $\overline{E}$. It follows that $E' = (\overline{E})'$
Lastly, E and E' do not always have the same limit points. Consider our construction in Exercise 5. E has three limit points while E' has no limit points.

□

7. Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space.

   (a) If $B_n = \bigcup_{i=1}^n A_i$, prove that $\overline{B_n} = \bigcup_{i=1}^n \overline{A_i}$ for $n \in \mathbb{N}$.
   Proof. If E is some subset of a metric space, then we denote by E' the set of limit points of E. We claim that

$$\left(\bigcup_{i=1}^n A_i\right)' = \bigcup_{i=1}^n A_i'$$

Call the left-hand side and right-hand side C and D, respectively. If $x \in D$, then x is a limit point of A_i for some $i \in \{1, 2, \ldots, n\}$. However, C includes the set of all limit points of the union of all A_i, which means that it includes the limit points for our specific A_i, hence $x \in C$. On the other hand, suppose that $x \in C$. Then for every neighborhood $N_r(x)$, we have $(\bigcup_{i=1}^n A_i) \cap (N_r(x) \setminus \{x\}) \neq \varnothing$. Consider the set

$$Q = \left\{ q \in \left(\bigcup_{i=1}^n A_i\right) \cap (N_{\frac{1}{k}}(x) \setminus \{x\}) \mid k \in \mathbb{N} \right\}$$

There must exist an A_i such that $A_i \cap Q$ is countably infinite. If not, then Q is finite and x is not a limit point. It follows that x is a limit point of A_i and hence $x \in D$. Thus, $C = D$
Observe that

$$\begin{aligned}
\overline{B_n} = B_n \cup B_n' &= \left(\bigcup_{i=1}^n A_i\right) \cup \left(\bigcup_{i=1}^n A_i\right)' \\
&= \left(\bigcup_{i=1}^n A_i\right) \cup \left(\bigcup_{i=1}^n A_i'\right) \\
&= \bigcup_{i=1}^n \overline{A_i}
\end{aligned}$$

Proof 2. Recall that the finite union of closed sets is closed. Since $\overline{A_i}$ is closed for all $i \in \{1, 2, \ldots n\}$, it follows that the finite union $\bigcup_{i=1}^n \overline{A_i}$ is closed. Further, it is clear that $B_n \subseteq \bigcup_{i=1}^n \overline{A_i}$. Hence, by Theorem 2.27, we have

$$\overline{B_n} \subseteq \bigcup_{i=1}^n \overline{A_i}$$

On the other hand, suppose that $x \in \bigcup_{i=1}^n \overline{A_i}$. Then there exists $\overline{A_i}$ such that $x \in \overline{A_i}$. But since $A_i \subseteq B_n$, we have $\overline{A_i} \subseteq \overline{B_n}$, hence $x \in \overline{B_n}$. The desired equality follows.

□

(b) If $B = \bigcup_{i=1}^\infty A_i$, prove that $\overline{B} \supseteq \bigcup_{i=1}^\infty \overline{A_i}$.
Proof. Let $C := \overline{B}$ and $D := \bigcup_{i=1}^\infty \overline{A_i}$. Suppose $x \in D$. Then $x \in \overline{A_i}$ for some $i \in \{1, 2, \ldots, n\}$. From Theorem 2.27, we know that $\overline{B}$ is closed, also, $A_i \subseteq B \subseteq \overline{B}$, hence by Theorem 2.27 again,

$$\overline{A_i} \subseteq \overline{B}$$

Thus, $x \in C$.

□

8. Is every point of every open set $E \subseteq \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.
Yes, every point of every open set is a limit point of E. This is not true for closed sets.
Proof. Let $E \subseteq \mathbb{R}^2$ be an open set. Let $x \in E$. Then, x is an interior point, which means there exists a neighborhood $N_r(x)$ for some $r > 0$ such that $N_r(x) \subseteq E$. For any neighborhood N, it is possible to pick an appropriate point $p \neq x$ such that $p \in N_r(x) \cap N$. Hence x is a limit point of E. Next, consider the closed set consisting of the point $(1, 1)$. Since it is a finite set, there are no limit points.

□

9. Let E° denote the set of all interior points of a set E. E° is called the interior of E.

(a) Prove that E° is always open.
Let $x \in E^\circ$. Then, x is an interior point of E, which means there exists a neighborhood $N_r(x) \subseteq E$. Recall that every neighborhood is an open set. Thus, for all $y \in N_r(x)$, there exists a $N_{r_2}(y) \subseteq N_r(x)$. But then y is an interior point of E. Hence, $y \in E^\circ$. This shows us that $N_r(x) \subseteq E^\circ$. Therefore, x is an interior point of E° and E° is open.

□

(b) Prove that E is open if and only if $E = E^\circ$.
Suppose E is open. By the definition of open, if $x \in E$, then x is an interior point. Hence $x \in E^\circ$. Suppose $x \in E^\circ$. Then there exists a neighborhood $N_r(x) \subseteq E$, namely, $x \in E$. So, $E = E^\circ$.
Now, suppose that $E = E^\circ$. So, if $x \in E$, then $x \in E^\circ$. Hence, E is open.

□

(c) If $G \subseteq E$ and G is open, prove that $G \subseteq E^\circ$.
Suppose that $x \in G$. Then there exists a neighborhood $N_r(x) \subseteq G$ since G is open. Since $G \subseteq E$, it follows that $N_r(x) \subseteq E$. Hence, x is an interior point of E, so $x \in E^\circ$. The conclusion follows.

$\square$

(d) Prove that the complement of E° is the closure of the complement of E.
We want to show that $(E^\circ)^c = \overline{E^c}$. By definition, $E^\circ \subseteq E$. Taking the complements, we have $(E^\circ)^c \supseteq E^c$. Further, since E° is open, $(E^\circ)^c$ is closed by Theorem 2.23. By Theorem 2.27, we have

$$(E^\circ)^c \supseteq \overline{E^c}$$

Now, suppose $x \in (E^\circ)^c$. Then, x is not an interior point of E. Suppose $x \in E^c$, it follows that $x \in \overline{E^c}$ and we are done. Suppose $x \in E$. Since x is not an interior point of E, this means that for every neighborhood N or x, $N \not\subseteq E$. So, there exists some $q \in N$ such that $q \notin E$. Thus, we may choose points $q_1, q_2, q_3, \ldots \in E^c$ (necessarily distinct from x) corresponding to neighborhoods $N_1(x), N_{\frac{1}{2}}(x), N_{\frac{1}{3}}(x), \ldots$. Hence, x is a limit point of E^c and $x \in \overline{E^c}$. Having shown the forward and reverse inclusion, we conclude that $(E^\circ)^c = \overline{E^c}$.

$\square$

(e) Do E and $\overline{E}$ always have the same interiors?
No. Consider $E = \mathbb{Q}$. Taking the closure, we have $\overline{E} = \mathbb{R}$. However

$$E^\circ = \mathbb{Q}^\circ = \varnothing \neq \mathbb{R} = (\overline{E})^\circ$$

(f) Do E and E° always have the same closures?
No. Consider $E = \mathbb{Z}$. Then $\overline{E} = \mathbb{Z}$ but $\overline{(E^\circ)} = \varnothing$.

10. Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p,q) = \begin{cases} 1 & \text{if } p \neq q \\ 0 & \text{if } p = q \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?
Proof. We verify the properties of a metric. By the definition of d, we see that $d(p,q) = 0$ if and only if $p = q$. Otherwise, if $p \neq q$, then $d(p,q) = 1 > 0$. So d is positive-definite. Also, $d(p,q) = d(q,p)$ since equality and non-equality are symmetric. To verify the triangle inequality, we must proceed by cases. Suppose that $p = q = r$. Then $d(p,r) = 0 = d(p,q) + d(q,r)$. Suppose that $p \neq q$ and $q = r$. Then $d(p,r) = 1 = d(p,q) + d(q,r)$. Suppose that $p = q$ and $q \neq r$. Then $d(p,r) = 1 = d(p,q) + d(q,r)$. Suppose that $p \neq q$ and $q \neq r$. Then $d(p,r) = 1 < d(p,q) + d(q,r)$. Hence d is a metric.

$\square$

11. This question is about checking if functions are metrics. It is answered in HW4

12. Let $K \subseteq \mathbb{R}$ consist of 0 and the numbers $\frac{1}{n}$ for $n \in \mathbb{N}$. Prove that K is compact directly from the definition (without using the Heine-Borel theorem).
Proof. Let $(V_\alpha)_{\alpha \in A}$ be an open cover of K. Since it is an open cover, we must have some open set V_1 such that $0 \in V_1$. Since $0 \in V_0$ and V_0 is open, 0 is an interior point. So there exists some neighborhood $N_r(0) \subseteq V_0$. If $r > 1$, then $\{V_0\}$ is a finite subcover of K. Now suppose $r \leq 1$. By the Archimedean property, there exists some $N \in \mathbb{N}$ such that $\frac{1}{N} < r$. Hence, for $n \geq N$, it follows that $\frac{1}{n} \in N_r(0)$ and so $\frac{1}{n} \in V_0$. There are now finitely many points $1, 2, \ldots, \frac{1}{N-1}$ left to cover. Since $(V_\alpha)_{\alpha \in A}$ is an open cover of K, we can find open sets $V_1, \ldots V_{N-1}$ such that $1 \in V_1, \frac{1}{2} \in V_2, \ldots, \frac{1}{N-1} \in V_{N-1}$. Therefore,

$$\{V_0, V_1, \ldots, V_{N-1}\}$$

is a finite, open subcover of K.

□

13. Construct a compact set of real numbers whose limit points form a countable set.
Since our set is compact, it must be closed and bounded. So, consider the set

$$\{0\} \cup \left\{ \frac{1}{n} + \frac{1}{m} : m, n \in \mathbb{N} \right\}$$

This set is bounded, with $|x| < 3$. It is closed because it contains all of its limit points. So this set is compact. There are a countable number of limit points since every $\frac{1}{n}$ is a limit point.

14. Give an example of an open cover of the segement $(0, 1)$ which has no finite subcover.
Consider the open cover

$$\left\{ \left(\frac{1}{n}, 1 \right) : n \in \mathbb{N} \text{ and } n > 1 \right\}$$

For any finite subcover, there is a smallest number $m \in \mathbb{N}$ such that the open set $(\frac{1}{m}, 1)$ is in the subcover. Then $\frac{1}{m+1}$ is an element of $(0, 1)$ which is not covered.

15. Show that Theorem 2.36 and its Corollary become false (in $\mathbb{R}$, for example) if the word compact is replaced by closed or by bounded.
Suppose that the word compact is replaced by closed. Then we may consider the sequence of closed sets

$$K_n = \{n, n+1, n+2, \ldots\}$$

It is clear that $K_n \supseteq K_{n+1}$ by construction. The sets are closed because none of the points are limit points. However,

$$\bigcap_{n=1}^{\infty} K_n = \varnothing$$

since no positive integer exists in this intersection. For any positive integer n, it does not belong to K_{n+1}. Since Corollary is implied by the theorem, its falsity means shows

the theorem is false as well.
Suppose that compact is replaced by bounded. Then the sequence of sets

$$K_n = \left(0, \frac{1}{n}\right)$$

satisfies the conditions of the hypothesis. However,

$$\bigcap_{n=1}^{\infty} K_n = \varnothing$$

16. Regard $\mathbb{Q}$, the set of all rational numbers, as a metric space, with $d(p, q) = |p - q|$. Let E be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show that E is closed and bounded in $\mathbb{Q}$, but that E is not compact. Is E open in $\mathbb{Q}$?
To show that E is bounded, notice that for all $x \in E$, we have $|x| < 3$. We now show E is closed. Let $x \in E$. Then we have $2 < x^2 < 3$ and $x \in \mathbb{Q}$. Considering the expression $x + \frac{1}{n}$, we want

$$\left(x + \frac{1}{n}\right)^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2} < 3$$
$$\frac{2x}{n} + \frac{1}{n^2} < 3 - x^2$$

By the Archimedean principle, we can find N_1 such that $\frac{2x}{N_1} < 3 - x^2 - 1$ and N_2 such that $\frac{1}{N_2^2} < 1$. Choose $N = \max N_1, N_2$. Then, the inequality above is satisfied. However, since it holds for N, it also holds for $n \geq N$. It follows that $x + \frac{1}{n} \in E$. Thus, E is closed since for any neighborhood of radius r, we can choose $\frac{1}{n}$ in a way that satisfies the above inequality while remaining in the neighborhood. Note that we do not need to check if any irrationals are limit points because $\mathbb{Q}$ is our metric space.
Consider the open cover

$$\left\{\mathbb{Q} \cap \left(\sqrt{2}, \sqrt{3} - \frac{1}{n}\right) : n \in \{2, 3, 4, \ldots\}\right\}$$

By the density of rationals, we can always find $x \in E$ such that $x \in \left(\sqrt{3} - \frac{1}{n}, \sqrt{3}\right)$, so any finite cover will not be a cover of E. Hence, E is not compact.
Lastly E is open in $\mathbb{Q}$ by Theorem 2.30 since we may write it as $\mathbb{Q} \cap (\sqrt{2}, \sqrt{3})$.

17. Let E be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0, 1]$? Is E compact? Is E perfect?
Suppose, for contradiction, that E is countable. We may mimic the construction in Theorem 2.14. Since E is countable, we may enumerate $x_1, x_2, \ldots \in E$. Further, each x_i has digits $x_{i,1}, x_{i,2}, x_{i,3}, \ldots$ We may construct a new $x_0 \in E$ such that $x_{0,j} = 4$ if $x_{j,j} = 7$ and vice versa. But then $x_0 \neq x_i$ for all $i \in \{1, 2, 3, \ldots\}$. ($\lightning$)
Hence E must be uncountable.
The set E is not dense in $[0, 1]$. Notice that the $\max(E) = \frac{7}{9}$ and $\min(E) = \frac{4}{9}$. Hence, between 0 and $\frac{4}{9}$ there is no member of E. Therefore, it is not dense in $[0, 1]$.

For the last two questions, we note that E is bounded since, for all $x \in E$, we have $|x| < 1$. Now, let $x \in E$. We show that x must be a limit point. Let $N_r(x)$ be an arbitrary neighborhood around x. Since $r > 0$, by the Archimedean principle, we can find $\frac{1}{10^n} < r$. We choose another element $q \in E$ such that $q_i = x_i$ for each digit, except at $n+1$. We flip the digit of x (from 4 to 7 or vice versa) at position $n+1$ for the digit of q at position $n+1$. Thus

$$|x - q| = \frac{3}{10^{n+1}} < \frac{1}{10^n} < r$$

So $q \in N_r(x) \cap E$ and $x \neq q$. Hence, x is a limit point since our choice of neighborhood was arbitrary. Since E is closed and bounded in $\mathbb{R}$, by Heine-Borel it is compact. Since E is closed and every point of E is a limit point, it is perfect.

18. Is there a nonempty perfect set in $\mathbb{R}^1$ which contains no rational number?
Solution. Yes. Let $\{r_1, r_2, \ldots\}$ be the set of all rational numbers in $E_0 = [\pi, \pi + 1]$, construct a subset of E_0 as below. Having chosen E_n $(n \geq 0)$ the subsets of E_0 such that E_n is a disjoint union of at most $2^{n+1} - 1$ intervals with irrational endpoints, each of positive length at most 3^{-n} and E_n does not contain r_j for $1 \leq j \leq n$. Define the set F_{n+1} obtained from E_n by removing the middle thirds of all intervals of E_n. Thus F_{n+1} is a disjoint union of at most $2^{n+2} - 2$ intervals with irrational endpoints (removing the middle thirds creates two intervals from each existing interval), each of positive length at most $3^{-(n+1)}$. If $r_{n+1} \notin F_{n+1}$, take $E_{n+1} = F_{n+1}$. If $r_{n+1} \in F_{n+1}$, then $r_{n+1} \in [a, b]$ for some interval $[a, b] \subset F_{n+1}$. Since a and b are irrational, we have $r_{n+1} \in (a, b)$. Let $\delta > 0$ be an irrational number such that $\delta < \min\{r_{n+1} - a, b - r_{n+1}\}$, and put

$$E_{n+1} = F_{n+1} - (r_{n+1} - \delta, r_{n+1} + \delta)$$

By doing this, we see that E_{n+1} is a disjoint union of at most $2^{n+2} - 1$ intervals with irrational endpoints, each of positive length at most $3^{-(n+1)}$ and E_{n+1} does not contain r_j for $1 \leq j \leq n+1$. Hence the resulting set $P = \bigcap_{n=0}^{\infty} E_n$ contains no rational number. Since E_n is closed for each $n \geq 0$, by Theorem 2.24(b), P is closed. Moreover, since $E_0 \supseteq E_1 \supseteq E_2 \cdots$, each E_n is bounded (by E_0), i.e. each E_n is actually a nonempty compact set. By Theorem 2.36, $P \neq \varnothing$. Finally, to show P is perfect, let $x \in P$, then $x \in E_n$ for each positive integer n. Let $x \in [a, b]$ for some interval $[a, b] \subseteq E_n$, noting that $b - a \leq 3^{-n}$. If $x \neq a$, then put $y = a$; If $x = a$, then put $y = b$. In both cases, we see that $y \in P$, $y \neq x$, and

$$|y - x| \leq b - a \leq 3^{-n} < 2 \cdot 3^{-n}$$

It follows that x is a limit point of P, and hence P is perfect.

□

19. (a) If A and B are disjoint, closed sets in some metric space X, prove that they are separated.
Since A and B are closed, we have $A = \overline{A}$ and $B = \overline{B}$. Then $A \cap \overline{B} = \varnothing$ and $\overline{A} \cap B = \varnothing$ since A and B are disjoint. So they are separated.

(b) Prove the same for disjoint open sets.
Suppose that A and B are disjoint open sets. Let $x \in A$. Since A is open, then x is an interior point, hence, there exists $r > 0$ such that $N_r(x) \subseteq A$. Further, x is not a limit point of B, since $N_r(x) \cap B = \varnothing$. Hence, $A \cap \overline{B} = \varnothing$. An analogous argument shows $\overline{A} \cap B = \varnothing$. Hence, A and B are separated.

(c) Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p,q) < \delta$, define B similarly, with $>$ in place of $<$. Prove that A and B are separated.
Notice that A is a neighborhood of radius δ around p, that is $A = N_\delta(p)$. Since it is a neighborhood, it is open. Also,

$$B = \{q \in X : d(p,q) > \delta\} = (\overline{A})^c$$

Hence, B is open. It is clear that A and B are disjoint. Applying (b), it follows that A and B are separated.

(d) Prove that every connected metric space with at least two points is uncountable. *Hint:* Use (c).
Let $x, y \in X$ and let $d(x,y) = d > 0$. Then, for every $\delta \in (0,d)$, there exists a point $z \in X$ such that $d(x,z) = \delta$. If not, we can define sets according to (c) which separate X. Hence, there is a subset of X that can be placed into a bijection with the interval $[0,d]$, and so X is uncountable.

20. Are closures and interiors of connected sets always connected? (Look at subsets of $\mathbb{R}^2$.)
Suppose X is a connected set. Consider its closure $\overline{X} = X \cup X'$, where X' is the set of limit points of X. Let $A, B \subseteq \overline{X}$ be two nonempty sets such that $A \cup B = \overline{X}$. Also, we have

$$(A \cap X) \cup (B \cap X) = X$$

Since X is connected, $(A \cap X)$ and $(B \cap X)$ must not be separated. So WLOG, there exists $x \in (A \cap X) \cap \overline{(B \cap X)} \subseteq A \cap \overline{B}$. The last equality follows since $A \cap X \subseteq A$ and $B \cap X$ is a subset of the closed set $\overline{B}$, so by Theorem 2.27, so is its closure. Hence, $\overline{X}$ is connected.
The interiors of connected sets may not be connected. Let E be the union of two closed disks which are tangent to each other at a point.

21. Let A and B be separated subsets of some $\mathbb{R}^k$, suppose $\mathbf{a} \in A$ and $\mathbf{b} \in B$, and define

$$\mathbf{p}(t) = (1-t)\mathbf{a} + t\mathbf{b}$$

for $t \in \mathbb{R}$. Put $A_0 = \mathbf{p}^{-1}(A), B_0 = \mathbf{p}^{-1}(B)$. Thus, $t \in A_0$ if and only if $\mathbf{p}(t) \in A$.

(a) Prove that A_0 and B_0 are separated subsets of $\mathbb{R}$.
Suppose, for contradiction, that A_0 and B_0 are not separated. Without loss of generality, suppose that $A_0 \cap \overline{B_0}$ is nonempty. Then, there exists a $t \in A_0 \cap \overline{B_0}$ satisfying $\mathbf{p}(t) \in A \cap \overline{B}$. Since $A \cap \overline{B}$ is nonempty, A and B are not separated. ($\lightning$)

(b) Prove that there exists $t_0 \in (0,1)$ such that $\mathbf{p}(t_0) \notin A \cup B$.
Suppose, for contradiction, that for all $t \in (0,1)$, we have $\mathbf{p}(t) \in A \cup B$. Hence $\mathbf{p}((0,1)) \subseteq A \cup B$. But then

$$A_0 \cup B_0 = \mathbf{p}^{-1}(A) \cup \mathbf{p}^{-1}(B) = \mathbf{p}^{-1}(A \cup B) \supseteq (0,1)$$

From part (a), we know that A_0 and B_0 are separated. This shows that $(0,1)$ is a union of two nonempty, separated sets. ($\lightning$)

(c) Prove that every convex subset of $\mathbb{R}^k$ is connected.
Recall that a subset E of $\mathbb{R}^k$ is called convex if for all $\mathbf{a}, \mathbf{b} \in E$ and $t \in (0,1)$, we have $(1-t)\mathbf{a}+t\mathbf{b} \in E$. We proved previously that if A and B are separated subsets of some E (the argument for $\mathbb{R}^k$ applies to any subset), then there exists some t_0 such that $(1-t_0)\mathbf{a} + t\mathbf{b} \notin E$, that is, E is not convex. By the contrapositive, if E is convex, then there do not exist separated subsets of E, meaning that E is connected.

22. A metric space is called *separable* if it contains a countable dense subset. Show that $\mathbb{R}^k$ is separable. *Hint:* Consider the set of points which have only rational coordinates.
Solution. The set $\mathbb{Q}^k$ is a countable, dense subset of $\mathbb{R}^k$. Since $\mathbb{Q}$ is countable, then $\mathbb{Q}^k$ is countable by Theorem 2.13: the finite Cartesian product of countable sets is countable. Pick an arbitrary $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$. To show that $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$, recall that $\mathbb{Q}$ is dense in $\mathbb{R}$. So each x_i is a limit point of $\mathbb{Q}$. Given an arbitrary neighborhood N of $\mathbf{x}$, we can find points q_i arbitrarily close to x_i, for $i \in \{1, 2, \ldots, k\}$. Thus, we can choose $\mathbf{q} = (q_1, \ldots, q_k) \in N$ distinct from $\mathbf{x}$.

23. A collection $\{V_\alpha\}$ of open subsets of X is said to be a *base* for X if the following is true: For every $x \in X$ and every open set $G \subseteq X$ such that $x \in G$, we have $x \in V_\alpha \subseteq G$ for some α. In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.
Prove that every separable metric space has a *countable* base. *Hint:* Take all neighborhoods with rational radius and center in some countable dense subset of X.
Solution. We use the collection suggested by the hint of all neighborhoods with rational radii and center in a given countable dense subset, D, of X. Let $x \in X$ and let G be an open set of X such that $x \in G$. Since G is open, there exists a neighborhood $N_r(x) \subseteq G$. Suppose $x \in D$. Then by the Archimedean property, we have $\frac{1}{n} < r$ for some $n \in \mathbb{N}$. Hence $N_{\frac{1}{n}}(x) \subset G$ and $N_{\frac{1}{n}}(x)$ is in our proposed base. Suppose $x \notin D$. Since D is dense, x must be a limit point of D. By the Archimedean property, we may again choose some $n \in \mathbb{N}$ so that $\frac{1}{n} < r$. Since x is a limit point of D, there exists $y \in N_{\frac{1}{2n}}(x) \cap D$. The neighborhood $N_{\frac{1}{2n}}(y)$ contains x since $d(x,y) < \frac{1}{2n}$. Further $N_{\frac{1}{2n}}(y) \subseteq G$ since it is contained in $N_r(x)$. Explicitly, for an arbitrary point $y' \in N_{\frac{1}{2n}}(y)$, we have

$$d(x,y') \leq d(x,y) + d(y,y') \leq \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n} < r$$

So $y' \in N_r(x)$. This shows that $N_{\frac{1}{2n}}(y)$ works as our V_α. Thus, the suggested collection of neighborhoods is indeed a countable base.

24. Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable. *Hint:* Fix $\delta > 0$, and pick $x_1 \in X$. Having chosen $x_1, \ldots, x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \ldots, j$. Show that this process must stop after a finite number of steps, and that X can therefore be covered by finitely many neighborhoods of radius δ. Take $\delta = \frac{1}{n}$, $n \in \mathbb{N}$, and consider the centers of the corresponding neighborhoods.

Proof. We continue from the setup given in the hint. Suppose for contradiction that the process never terminates and consider the sequence $\{x_n\}$ resulting from this construction. By construction, we have $d(x_i, x_j) \geq \delta$ for all distinct $i, j \in \mathbb{N}$. As a result, every x_n is an isolated point. But then $\{x_n\}$ is an infinite subset with no limit points ($\lightning$). Thus, the process terminates.

For the same reason, X must be a bounded set. If X were not bounded, then for all real numbers M and for all $q \in X$ there exists $p \in E$ such that $d(p, q) \geq M$. Select an arbitrary point $x_1 \in X$. Assume $x_1, \ldots, x_{j-1}$ are chosen already so that $d(x_i, x_j) \geq \delta$ for $i = 1, \ldots, j-1$. Since X is not bounded, we may choose $x_{j+1} \in X$ so that

$$d(x_1, x_{j+1}) \geq \delta + \max\{d(x_1, x_i) : i = 1, \ldots, j\}$$

Using the triangle inequality, we see that

$$\begin{aligned} d(x_i, x_{j+1}) &\geq d(x_{j+1}, x_1) - d(x_1, x_i) \\ &\geq \delta + \max\{d(x_1, x_i) : i = 1, \ldots, j\} - d(x_1, x_i) \\ &\geq \delta \end{aligned}$$

Hence, if X is unbounded, the process does not terminate. So X must be bounded. Because X is bounded, it is possible to cover by finitely many neighborhoods of radius δ. Take $\delta = \frac{1}{n}$ and let E_n be the finite set of centers of neighborhoods with radius $\frac{1}{n}$ that cover X. We claim that

$$D = \bigcup_{n=1}^{\infty} E_n$$

is a countable and dense subset of X. As the countable union of sets that are at most countable, D is countable by the Corollary to Theorem 2.12. Now, consider $x \in X$. If $x \in D$, then we are done. Suppose $x \notin D$ and let $\varepsilon > 0$ be arbitrary. By the Archimedean property, we can find $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. Since X is covered by the union of neighborhoods of radius $\frac{1}{N}$ around centers in E_N, there exists some $e \in E_N$ such that $d(x, e) < \frac{1}{N}$. Hence, x is a limit point of D and D is dense in X. Therefore, X is separable.

□

25. Prove that every compact metric space K has a countable base, and that K is therefore separable. *Hint:* For every positive integer n, there are finitely many neighborhoods of radius $\frac{1}{n}$ whose union covers K.

Proof. By Theorem 2.37, if E is an infinite subset of a compact set K, then E has a limit point in K. Hence, K is a metric space in which every infinite subset has a limit point. By Problem 24, we see that K is separable.

□

26. Let X be a metric space in which every infinite subset has a limit point. Prove that X is compact. *Hint:* By Exercises 23 and 24, X has a countable base. It follows that every open cover of X has a *countable* subcover $\{G_n\}$, $n = 1, 2, 3, \ldots$ If no finite subcollection of $\{G_n\}$ covers X, then the complement F_n of $G_1 \cup \ldots \cup G_n$ is nonempty for each n, but $\bigcap F_n$ is empty. If E is a set which contains a point from each F_n, consider limit point of E, and obtain a contradiction.
Proof. We follow the outline of the hint. From Exercise 24, we know that X is a separable metric space. From Exercise 23, we know that every separable metric space has a countable base. Hence, X has a countable base, $\{G_n\}$. Since this countable base covers X, we can use it as a countable subcover of any open cover of X.
Suppose, for contradiction, that no finite subcollection of $\{G_n\}$ covers X. Then the complement F_n of $G_1 \cup \ldots \cup G_n$ is nonempty for each n, but $\bigcap F_n$ is empty since

$$\bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} (G_1 \cup \ldots \cup G_n)^c = \left(\bigcup_{n=1}^{\infty} G_n \right)^c = X^c$$

Let E be a set which contains a point from each F_n. Then E is an infinite set (if E is finite, then the intersection of F_n is nonempty) and E has a limit point. But since F_n is the complement of an open set, it is closed. The infinite intersection of closed sets is closed, so $\bigcap F_n$ is closed. But since it is closed, it must contain all of its limit points, so $\bigcap F_n$ is not empty ($\lightning$). Therefore, we conclude that X is compact.

$\square$

27. Define a point p in a metric space X to be a *condensation point* of a set $E \subseteq X$ if every neighborhood of p contains uncountably many points of E.
Suppose $E \subseteq \mathbb{R}^k$, E is uncountable, and let P be the set of all condensation points of E. Prove that P is perfect and that at most countably many points of E are not in P. In other words, show that $P^c \cap E$ is at most countable. *Hint.* Let $\{V_n\}$ be a countable base of $\mathbb{R}^k$, let W be the union of those V_n for which $E \cap V_n$ is at most countable, and show that $P = W^c$.
Proof.

(a) Consider the outline and notation provided in the hint. Suppose $x \in W$. Then $x \in V_k$ for some $V_k \in \{V_n\}$. Since V_k is open, there exists a neighborhood N such that $x \in N \subseteq V_k$. Since $E \cap V_n$ is at most countable, $E \cap N$ must also be at most countable since it is a subset. Hence $x \in P^c$ since it is not a condensation point. On the other hand, suppose that $x \in P^c$. Then there exists a neighborhood N of x that contains at most countably many points of E. Since $\{V_n\}$ is a countable base of $\mathbb{R}^k$, there exists some V_k satisfying $x \in V_k \subseteq N$. It follows that $V_k \cap E$ is at most countable and $x \in W$. We have shown that $W = P^c$. By taking complements, we arrive at the conclusion that $P = W^c$.

(b) We show that $P^c \cap E$ is at most countable. Consider that

$$P^c \cap E = W \cap E = \left(\bigcup V_n \right) \cap E = \bigcup V_n \cap E$$

Thus, $P^c \cap E$ is the countable union of sets that are at most countable. This union is countable by the Corollary to Theorem 2.12

(c) Now, we show that P is perfect. Since W is the union of open sets V_n, this union is open. Hence, its complement is closed. Suppose, for contradiction, that $x \in P$ and x is not a limit point of P. Since x is not a limit point of P, there exists a neighborhood $N_r(x)$ such that $N_r(x) \cap P = \{x\}$. Since $x \in P$, then $N_r(x) \cap E$ has uncountably many points. But then uncountably many points in $N_r(x)$ are in E but not in P, which contradicts (b) ($\lightning$). We have shown that $x \in P$ if and only if x is a limit point of P. Therefore, P is perfect.

$\square$

28. Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable. (*Corollary:* Every countable closed set in $\mathbb{R}^k$ has isolated points). *Hint:* Use Exercise 27.
Proof. Suppose that X is our separable metric space. By Problem 23, every separable metric space has a countable base. Let E be a closed set. Suppose that E is at most countable. Then notice that $E = \varnothing \cup E$, so that it is a union of a perfect set and a set which is at most countable. Now, suppose that E is uncountable. Let P be the set of all condensation points of E. Notice that if $x \in P$, then $x \in E$, since a condensation point of E is a limit point of E, and E is closed. Hence, $P \subseteq E$. Also, using the construction in Exercise 27, we have

$$E = (E \cap P) \cup (E \cap W) = P \cup (E \cap W) = P \cup \left(E \cap \bigcup V_n\right)$$

This shows that E is again the union of a perfect set and a set which is at most countable.

$\square$

29. Prove that every open set in $\mathbb{R}^1$ is the union of an at most countable collection of disjoint segments. *Hint:* Use Exercise 22.
Proof. If U is empty, then it is the union of a finite collection of disjoint segments, namely the empty collection. Let U be a nonempty open subset of $\mathbb{R}$. For each $x \in U$, define

$$a_x = \inf\{a : [a, x] \subseteq U\}$$
$$b_x = \sup\{b : [x, b] \subseteq U\}$$

where we note that a_x is possibly $-\infty$ and b_x is possibly ∞. We claim that

$$U = \bigcup_{x \in U} (a_x, b_x)$$

It is clear that U is a subset of this union. Suppose that $y \in \bigcup_{x \in U}(a_x, b_x)$. Then $y \in (a_x, b_x)$ for some $x \in U$. Since $y \in (a_x, b_x)$, we know that $a_x < y < b_x$. Since a_x is an infimum, there exists $a' \in \{a : [a, x] \subseteq U\}$ satisfying $a_x < a' < y$. Similarly, there exists b' satisfying $y < b' < b_x$. Hence $y \in [a', b'] \subseteq U$ so that $y \in U$. This shows that U is the union of segments.
Now, we show that the segments are disjoint, that is if $x, y \in U$, then $(a_x, b_x) = (a_y, b_y)$

or $(a_x, b_x) \cap (a_y, b_y) = \varnothing$. Suppose, for contradiction, that there exists $x, y \in U$ such that $(a_x, b_x) \neq (a_y, b_y)$ and $(a_x, b_x) \cap (a_y, b_y) \neq \varnothing$. Let $z \in (a_x, b_x) \cap (a_y, b_y)$. Since $z \in (a_x, b_x)$, it follows that $a_x = \inf\{a : [a, z] \subseteq U\}$. The same reasoning shows $a_y = \inf\{a : [a, z] \subseteq U\}$. Hence $a_x = a_y$. An analogous argument shows $b_x = b_y$. But then $(a_x, b_x) = (a_y, b_y)$, a contradiction.
Finally, this union of sets is at most countable because a rational number exists in every interval. Hence, the intervals in the union may be indexed by a subset of the rational numbers.

□

30. Imitate the proof of Theorem 2.43 to obtain the following result:

    If $\mathbb{R}^k = \bigcup_{n=1}^{\infty} F_n$, where each F_n is a closed subset of $\mathbb{R}^k$, then at least one F_n has a nonempty interior.

    Equivalent statement: If G_n is a dense open subset of $\mathbb{R}^k$, for $n = 1, 2, 3, \dots$ then $\bigcap_{n=1}^{\infty} G_n$ is not empty (in fact, it is dense in $\mathbb{R}^k$).

(This is a special case of Baire's theorem; see Exercise 22, Chap. 3, for the general case.)
Proof. First, we prove the second statement. Let $x \in \mathbb{R}^k$ and $\varepsilon > 0$. Since G_n are dense open subsets of $\mathbb{R}^k$, we can choose a $g_1 \in G_1$ and $g_1 \in N_\varepsilon(x)$. Moreover, there exists $\delta_1 > 0$ such that $\overline{N_{\delta_1}(g_1)} \subseteq N_\varepsilon(x) \cap G_1$. Next, choose $g_2 \in G_2$ such that $g_2 \in N_{\delta_1}(g_1)$ and choose $\delta_2 > 0$ such that $\overline{N_{\delta_2}(g_2)} \subseteq N_{\delta_1}(g_1) \cap G_2$. Continue this process, the resulting set

$$\bigcap_{n=1}^{\infty} \overline{N_{\delta_n}(g_n)}$$

contains a point $g \in \mathbb{R}^k$ by the corollary to Theorem 2.36. Clearly, $g \in G_n$ for $n = 1, 2, \dots$, and so $g \in \bigcap_{n=1}^{\infty} G_n$. Also, $g \in \overline{N_{\delta_1}(g_1)} \subseteq N_\varepsilon(x)$. Thus $\bigcap_{n=1}^{\infty} G_n$ is dense in $\mathbb{R}^k$.
All the sets G_n are dense in $\mathbb{R}^k$. Note that we can choose such a g_2 because G_2 is dense in $\mathbb{R}^k$. To be explicit, g_1 is a limit point of G_2, so we can find a point in G_2 arbitrarily close to g_1 (i.e. less than δ_1 away).
Next, we prove the equivalence of the two statements. Suppose the first statement holds, let G_n be a dense open subset of $\mathbb{R}^k$ for $n = 1, 2, \dots$ Denote the closed set $F_n = (G_n)^c$ for each n. Since G_n is dense in $\mathbb{R}^k$, for $x \in F_n$ and for $r > 0$, we have $N_r(x) \cap G_n \neq \varnothing$. This means F_n has an empty interior for each n. Using the first statement and De Morgan's Law, we have

$$\left(\bigcap_{n=1}^{\infty} G_n\right)^c = \bigcup_{n=1}^{\infty} F_n \neq \mathbb{R}^k$$

This shows that $\bigcap_{n=1}^{\infty} G_n \neq \varnothing$. Conversely, suppose the second statement holds, let F_n be a closed set for $1, 2, \dots$ such that $\mathbb{R}^k = \bigcup_{n=1}^{\infty} F_n$. Denote the open set $G_n = (F_n)^c$ for each n. Using De Morgan's law again

$$\varnothing = \left(\bigcup_{n=1}^{\infty} F_n\right)^c = \bigcap_{n=1}^{\infty} G_n$$

It follows by the contrapositive of the second statement that there is an integer N such that G_N is not dense in $\mathbb{R}^k$, that is, there exists $x \in \mathbb{R}^k$ and $r > 0$ such that $N_r(x) \cap G_N = \varnothing$. But then $N_r(x) \subseteq (G_N)^c = F_N$ for some N.

$\square$

One detail regarding the last inclusion, recall that

$$N_r(x) = (N_r(x) \cap G_N) \cup (N_r(x) \cap G_N^c) = N_r(x) \cap G_N^c \subseteq G_N^c$$

Chapter 3: Numerical Sequences and Series

1. Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?
   Proof. Suppose that $s_n \to s$. Let $\varepsilon > 0$ be arbitrary. There exists an N such that for all $n \leq N$ we have
   $$||s_n| - |s|| \leq |s_n - s| < \varepsilon$$
   where the first inequality follows from the reverse triangle inequality. Thus $|s_n| \to |s|$.
   $\square$

   The converse is not true. Consider that the sequence with terms $|a_n| = |(-1)^n|$ converges but the sequence $\{-1, 1, -1, 1\ldots\}$ clearly does not.

2. Calculate $\lim_{n\to\infty}(\sqrt{n^2+n} - n)$.
   Consider
   $$\begin{aligned}\sqrt{n^2+n} - n = \sqrt{n^2+n} - n \cdot \frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} &= \frac{n}{\sqrt{n^2+n}+n} \\ = \frac{n}{\sqrt{n^2+n}+n} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} &= \frac{1}{\sqrt{1+\frac{1}{n}}+1}\end{aligned}$$
   Taking the limit, we see that
   $$\lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2}$$

3. If $s_1 = \sqrt{2}$, and
   $$s_{n+1} = \sqrt{2+\sqrt{s_n}} \quad (n = 1, 2, 3, \ldots)$$
   prove that $\{s_n\}$ converges, and that $s_n < 2$ for $n = 1, 2, 3, \ldots$.
   Proof. First, notice that
   $$2 = \sqrt{2}\sqrt{2} > \sqrt{2} \cdot 1 = s_1$$
   Suppose that $s_k < 2$ for some $k \in \mathbb{N}$. Then, since $\sqrt{2} > \sqrt{s_k}$, we have
   $$2 = \sqrt{2+2} > \sqrt{2+\sqrt{s_k}} = s_{k+1}$$
   Thus, $s_n < 2$ for all $n \in \mathbb{N}$ holds by induction. Further, observe that $\{s_n\}$ is monotonically increasing. We show this via induction as well. For the base case, we have
   $$s_1 = \sqrt{2} < \sqrt{2+\sqrt{s_1}} = s_2$$
   Assume that $s_k > s_{k-1}$. Then we see
   $$s_k = \sqrt{2+\sqrt{s_{k-1}}} < \sqrt{2+\sqrt{s_k}} = s_{k+1}$$
   Moreover, it is clear that $s_n > 0$ for all $n \in \mathbb{N}$. Hence $\{s_n\}$ is bounded and monotone. Therefore, it converges by Theorem 3.14
   $\square$

4. Find the upper and lower limits of the sequence $\{s_n\}$ defined by

$$s_1 = 0; \quad s_{2m} = \frac{s_{2m-1}}{2}; \quad s_{2m+1} = \frac{1}{2} + s_{2m}$$

We compute the first few terms of the sequence manually

$$0, 0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4} + \frac{1}{2}, \frac{\frac{1}{4} + \frac{1}{2}}{2}, \frac{\frac{1}{4} + \frac{1}{2}}{2} + \frac{1}{2}, \frac{\frac{\frac{1}{4}+\frac{1}{2}}{2} + \frac{1}{2}}{2}$$

Simplifying

$$0, 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{3}{8}, \frac{7}{8}, \frac{7}{16}, \ldots$$

Thus, it appears that the lower limit of $\{s_n\}$ is $\frac{1}{2}$ while the upper limit is 1.

5. For any two real sequences $\{a_n\}, \{b_n\}$, prove that

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

provided the sum on the right is not of the form $\infty - \infty$.
Proof. Suppose that $\limsup_{n\to\infty} a_n = +\infty$. From our assumption, we must have $\limsup_{n\to\infty} b_n = +\infty$ or $\limsup_{n\to\infty} b_n = b$ for some $b \in \mathbb{R}$. In either case, the desired inequality holds. In the cases where $\limsup_{n\to\infty} b_n = +\infty$, $\limsup_{n\to\infty} a_n = -\infty$, or $\limsup_{n\to\infty} b_n = -\infty$, similar arguments apply. Now, the remaining case is where

$$\limsup_{n\to\infty} a_n = a, \quad \limsup_{n\to\infty} b_n = b$$

for $a, b \in \mathbb{R}$. Let $s_n = a_n + b_n$. We claim that

$$s_n < a + b + \varepsilon$$

for all $\varepsilon > 0$. By Theorem 3.17(b), we may choose N_1 for which $a_n < a + \frac{\varepsilon}{2}$ for all $n \geq N_1$; likewise, we may choose N_2 for which $b_n < b + \frac{\varepsilon}{2}$ for all $n \geq N_2$. Hence, for $N = \max\{N_1, N_2\}$, we have

$$s_n < a + b + \varepsilon$$

for all $n \geq N$. It follows that $s_n \leq a + b$. Moreover, for any subsequence $s_{n_i} \to s$, we have $s_{n_i} \leq a + b$ for all $n_i \geq N$. Hence $s \leq a + b$. Since $a + b$ is an upper bound on the set of all subsequential limits, the desired inequality follows.

$\square$

6. Investigate the behavior (convergence or divergence) of Σa_n if

   (a) $a_n = \sqrt{n+1} - \sqrt{n}$;
   Observe that

   $$s_n = \sum_{k=1}^{n} \sqrt{k+1} - \sqrt{k} = \sqrt{n+1} - 1$$

   Since $\lim_{n\to\infty} \sqrt{n+1}$ diverges, we see that $\sum a_n$ diverges as well.

(b) $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}$;
Observe that

$$\frac{\sqrt{n+1}-\sqrt{n}}{n} = \frac{1}{n(\sqrt{n+1}+\sqrt{n})} < \frac{1}{n^{\frac{3}{2}}}$$

Since the series associated with terms $\frac{1}{n^{\frac{3}{2}}}$ converges by the p-test, it follows that $\sum a_n$ converges by the comparison test.

(c) $a_n = (\sqrt[n]{n}-1)^n$; Observe that

$$\limsup_{n\to\infty}(\sqrt[n]{n}-1) = 0$$

since, by Theorem 3.20, $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$. Also, by the Root Test (Theorem 3.33), since $\limsup_{n\to\infty}(\sqrt[n]{n}-1) < 0$, this series converges.

(d) $a_n = \frac{1}{1+z^n}$, for complex values of z.
If $|z| \leq 1$, then we have $|a_n| \geq \frac{1}{2}$ for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} a_n \neq 0$, it follows that $\sum a_n$ does not converge for these values of z. If $|z| > 1$, then $\frac{1}{|z|^n}$ converges since it is a geometric series with $r = \frac{1}{|z|} < 1$. Hence, by the comparison test $\sum a_n$ converges as well.

7. Prove that the convergence of Σa_n implies the convergence of

$$\sum \frac{\sqrt{a_n}}{n}$$

if $a_n \geq 0$.
Proof. Suppose that $\sum a_n = A$. By Theorem 3.28, we know that $\sum \frac{1}{n^2}$ converges, call its limit B. Since $a_n \geq 0$, it follows that $\sum \frac{\sqrt{a_n}}{n}$ is monotonic. By Cauchy-Schwarz, we have

$$\sum_{i=0}^{n} \frac{\sqrt{a_i}}{i} = \sum_{i=0}^{n} \sqrt{a_i} \cdot \sqrt{\frac{1}{i^2}} \leq \left(\sum_{i=0}^{n} a_i\right)^{\frac{1}{2}} \left(\sum_{i=0}^{n} \frac{1}{i^2}\right)^{\frac{1}{2}} < \sqrt{A \cdot B}$$

Hence, $\sum \frac{\sqrt{a_n}}{n}$ is bounded and so it converges by Theorem 3.14.

$\square$

8. If $\sum a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges.
Proof. Since $\{b_n\}$ is monotonic and bounded, then it also converges by Theorem 3.14. Say $b_n \to b$. Let $\varepsilon > 0$ be arbitrary. Since $\sum a_n$ converges, it is Cauchy and there exists $N_1 \in \mathbb{N}$ such that

$$\left|\sum_{k=n}^{m} a_n\right| \leq \frac{\varepsilon}{|b| + \varepsilon}$$

for $n \geq m \geq N_1$.
Moreover, we can choose $N_2 \in \mathbb{N}$ such that

$$|b_n - b| < \varepsilon \implies |b_n| < |b| + \varepsilon$$

for $n \geq N_2$. Now, take $N = \max\{N_1, N_2\}$. Multiplying, we have

$$\left|\sum_{k=n}^{m} a_n b_n\right| < \left|\sum_{k=n}^{m} a_n(|b| + \varepsilon)\right| \leq \varepsilon$$

for $n \geq m \geq N$. Therefore, $\sum a_n b_n$ converges by Theorem 3.22.

$\square$

9. Find the radius of convergence of each of the following power series.

(a)

$$\sum n^3 z^n$$

We consider $\limsup_{n\to\infty} \sqrt[n]{n^3} = 1$. Hence, $R = 1$.

(b)

$$\sum \frac{2^n}{n!} z^n$$

We consider the ratio

$$\left|\frac{\frac{2^{n+1}}{(n+1)!} z^{n+1}}{\frac{2^n}{n!} z^n}\right| = \left|\frac{2z}{n+1}\right|$$

It is clear that

$$\limsup_{n\to\infty} \frac{2}{n+1} = 0$$

Hence, for any complex z, $\limsup_{n\to\infty} |\frac{2z}{n+1}| = 0$ we have convergence by Theorem 3.34. This means that the radius of convergence is infinite.

(c)

$$\sum \frac{2^n}{n^2} z^n$$

Observe

$$\limsup_{n\to\infty} \sqrt[n]{\frac{2^n}{n^2}} = \lim_{n\to\infty} \frac{2}{\sqrt[n]{n^2}} = 2$$

Hence, the radius of convergence is $R = \frac{1}{2}$.

(d)

$$\sum \frac{n^3}{3^n} z^n$$

Observe

$$\limsup_{n\to\infty} \sqrt[n]{\frac{n^3}{3^n}} = \limsup_{n\to\infty} \frac{\sqrt[n]{n^3}}{3} = \frac{1}{3}$$

Hence, the radius of convergence is $R = 3$.

10. Suppose that the coefficients of the power series $\sum a_n z^n$ are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.
Proof. Let $\{a_n\}$ be an arbitrary sequence of integers with infinitely many nonzero terms. It follows that

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|} \geq \limsup_{n\to\infty} \sqrt[n]{1} = 1$$

Hence, we see that for the radius of convergence

$$R = \frac{1}{\alpha} \leq 1$$

□

11. Suppose $a_n > 0$, $s_n = a_1 + \ldots + a_n$ and $\sum a_n$ diverges.

(a) Prove that $\sum \frac{a_n}{1+a_n}$ diverges.
Proof. Suppose that $\sum a_n$ diverges. If there exists an $N \in \mathbb{N}$ such that $a_n \leq 1$ for $n \geq N$, then

$$\frac{a_n}{1+a_n} \geq \frac{a_n}{2}$$

So that $\sum \frac{a_n}{1+a_n}$ diverges by comparison to $\sum a_n$ (Theorem 3.25). On the other hand, if $a_n > 1$ for infinitely many $n \in \mathbb{N}$, then $2a_n > a_n + 1$. Hence $\frac{a_n}{a_n+1} > \frac{1}{2}$. It is clear that $\lim_{n\to\infty} \frac{a_n}{a_n+1} \neq 0$, so the associated series diverges.

□

(b) Prove that

$$\frac{a_{N+1}}{s_{N+1}} + \ldots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$$

and deduce that $\sum \frac{a_n}{s_n}$ diverges.
Proof. Since $a_n > 0$, it follows that $s_{n+1} \geq s_n$ for all $n \in \mathbb{N}$. Hence $\frac{a_{N+j}}{s_{N+j}} \geq \frac{a_{N+j}}{s_{N+k}}$ for all natural numbers k, j such that $k \geq j$. Observe that

$$\frac{a_{N+1}}{s_{N+1}} + \ldots + \frac{a_{N+k}}{s_{N+k}} \geq \frac{a_{N+1} + a_{N+2} + a_{N+k}}{s_{N+k}} = \frac{s_{N+k} - s_N}{s_{N+k}} = 1 - \frac{s_N}{s_{N+k}}$$

Since s_n is monotonic and diverges, it is not bounded. Hence, for some fixed $N \in \mathbb{N}$

$$\lim_{k\to\infty} \frac{s_N}{s_{N+k}} = 0$$

Choose $\varepsilon = \frac{1}{2}$. Let $N \in \mathbb{N}$ be arbitrary. We can choose $k \in \mathbb{N}$ such that $\frac{s_N}{s_{N+k}} < \frac{1}{2}$. Hence

$$\frac{a_{N+1}}{s_{N+1}} + \ldots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}} > 1 - \frac{1}{2} = \varepsilon$$

Therefore, $\sum \frac{a_n}{s_n}$ is not Cauchy and so it diverges.

□

(c) Prove that

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and deduce that $\sum \frac{a_n}{s_n^2}$ converges.

Proof. Observe that

$$\frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{s_n - s_{n-1}}{s_{n-1}s_n} = \frac{a_n}{s_{n-1}s_n} > \frac{a_n}{s_n^2}$$

Notice that

$$\sum_{n=2}^{N} \frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{1}{s_1} - \frac{1}{s_N}$$

Since s_n diverges and is unbounded, we see that

$$\sum_{n=2}^{\infty} \frac{1}{s_{n-1}} - \frac{1}{s_n} = \lim_{n\to\infty} \frac{1}{s_1} - \frac{1}{s_n} = \frac{1}{a_1}$$

Now, $\sum \frac{a_n}{s_n^2}$ converges by the comparison test.

□

(d) What can be said about

$$\sum \frac{a_n}{1+na_n} \text{ and } \sum \frac{a_n}{1+n^2a_n}$$

Observe that

$$\frac{a_n}{1+n^2a_n} = \frac{1}{\frac{1}{a_n}+n^2} < \frac{1}{n^2}$$

Hence this series converges by the comparison test. On the other hand, $\sum \frac{a_n}{1+na_n}$ may converge or diverge based on the series. For instance, if $a_n = 1$, we have the harmonic series. Alternatively if $a_n = \frac{1}{n^2}$, we have

$$\frac{1}{n^2} \cdot \frac{1}{1+\frac{1}{n}} < \frac{1}{n^2}$$

which converges by comparison.

12. Suppose $a_n > 0$ and $\sum a_n$ converges. Put

$$r_n = \sum_{m=n}^{\infty} a_m$$

(a) Prove that

$$\frac{a_m}{r_m} + \ldots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

if $m < n$, and deduce that $\sum \frac{a_n}{r_n}$ diverges.

Proof. Since $\sum a_n$ converges, r_n and r_m converge as well. Notice that $r_n > r_{n+1}$ for all $n \in \mathbb{N}$. Hence

$$\frac{a_m}{r_m} + \ldots + \frac{a_n}{r_n} > \frac{a_m + \ldots + a_n}{r_m} = \frac{r_m - r_n}{r_m} = 1 - \frac{r_n}{r_m}$$

Since $r_m > r_n$, we see that $1 - \frac{r_n}{r_m} > 0$.
Let $m \in \mathbb{N}$ be arbitrary. Choose any $n > m$. Then, we know that $1 - \frac{r_n}{r_m} > 0$. Choose ε such that $1 - \frac{r_n}{r_m} > \varepsilon > 0$. It follows that

$$\frac{a_m}{r_m} + \ldots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m} > \varepsilon$$

showing that $\sum \frac{a_n}{r_n}$ is not Cauchy. Therefore, it diverges.

□

(b) Prove that

$$\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$$

and deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.
Proof. Since $r_n > r_{n+1}$, it follows that $\sqrt{r_n} > \sqrt{r_{n+1}}$. Hence

$$\frac{a_n}{2\sqrt{r_n}} < \frac{a_n}{\sqrt{r_n} + \sqrt{r_{n+1}}} = \frac{r_n - r_{n+1}}{\sqrt{r_n} + \sqrt{r_{n+1}}} = \sqrt{r_n} - \sqrt{r_{n+1}}$$

from which the desired inequality follows. First, $\sum \frac{a_n}{\sqrt{r_n}}$ is monotonic. Further

$$\sum_{n=1}^{N} 2(\sqrt{r_n} - \sqrt{r_{n+1}}) = 2(\sqrt{r_1} - \sqrt{r_{N+1}}) < 2\sqrt{r_1}$$

Thus, $\sum \frac{a_n}{\sqrt{r_n}}$ is bounded and converges by Theorem 3.14 (MCT).

□

13. Prove that the Cauchy product of two absolutely convergent series converges absolutely.
Proof. Let $\sum a_n$ and $\sum b_n$ be two absolutely convergent series. By definition, this means $\sum |a_n|$ and $\sum |b_n|$ converge. Let $\sum |a_n| = A$ and $\sum |b_n| = B$. By Theorem 3.50, we see that the Cauchy product of these two series converges and

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} |a_k b_{n-k}| = AB$$

By the triangle inequality, we know that

$$\left| \sum_{k=0}^{n} a_k b_{n-k} \right| \le \sum_{k=0}^{n} |a_k b_{n-k}|$$

Hence, the Cauchy product of $\sum a_n$ and $\sum b_n$ converges absolutely by comparison.

□

14. If $\{s_n\}$ is a complex sequence, define its arithmetic means σ_n by

$$\sigma_n = \frac{s_0 + s_1 + \ldots + s_n}{n+1} \qquad (n = 0, 1, 2, \ldots)$$

(a) If $\lim s_n = s$, prove that $\lim \sigma_n = s$.
Proof. Let $\varepsilon > 0$ be arbitrary. Choose $N_1 \in \mathbb{N}$ such that $|s_n - s| < \frac{\varepsilon}{2}$ for all $n \geq N_1$. Put $c = s_0 + s_1 + \ldots + s_{N_1-1} - N_1 s$. We can choose $N_2 \in \mathbb{N}$ such that $\frac{c}{N_2} < \frac{\varepsilon}{2}$. Put $N = \max\{N_1, N_2\}$. For $n \geq N$ we have

$$\begin{aligned}
|\sigma_n - s| &= \left| \frac{s_0 + s_1 + \ldots + s_n}{n+1} - \frac{(n+1)s}{n+1} \right| \\
&= \left| \frac{s_0 + \ldots + s_{N_1-1} - N_1 s}{n+1} + \frac{s_{N_1} - s}{n+1} + \ldots + \frac{s_n - s}{n+1} \right| \\
&< \left| \frac{s_0 + \ldots + s_{N_1-1} - N_1 s}{N} \right| + \left| \frac{s_{N_1} - s}{n+1} \right| + \ldots + \left| \frac{s_n - s}{n+1} \right| \\
&< \frac{\varepsilon}{2} + \left(\frac{n - N_1 + 1}{n+1} \right) \cdot \frac{\varepsilon}{2} < \varepsilon
\end{aligned}$$

□

(b) Construct a sequence $\{s_n\}$ which does not converge although $\lim \sigma_n = 0$.
Simply consider the sequence $1, -1, 1, -1, \ldots$. The sequence does not converge because there are two distinct subsequential limits. However, $\lim \sigma_n$ converges to 0 since we can just pick $N \in \mathbb{N}$ so that $\frac{1}{N} < \varepsilon$.

(c) Can it happen that $s_n > 0$ for all n and that $\limsup_{n\to\infty} s_n = \infty$, although $\lim \sigma_n = 0$.
Let $a_n = \frac{1}{2^n}$ and $b_n = \begin{cases} k & \text{if } n = 2^k \\ 0 & \text{otherwise} \end{cases}$ and put $s_n = a_n + b_n$. Then, we have

$$\sigma_n = \frac{1}{n+1} \sum_{i=0}^{n} \frac{1}{2^i} + \frac{1}{n+1} \sum_{i=0}^{\lfloor \log_2(n) \rfloor} i < \frac{2}{n} + \frac{\log_2(n)^2}{n}$$

Since the right hand side converges to zero, it follows that $\lim \sigma_n = 0$.

(d) Put $a_n = s_n - s_{n-1}$, for $n \geq 1$. Show that

$$s_n - \sigma_n = \frac{1}{n+1} \sum_{k=1}^{n} k a_k$$

Assume that $\lim(n a_n) = 0$ and that $\{\sigma_n\}$ converges. Prove that $\{s_n\}$ converges. [This gives a converse of (a), but under the additional assumption that $n a_n \to 0$.]
Proof. Observe that

$$\begin{aligned}
\frac{1}{n+1} \sum_{k=1}^{n} k a_k &= \frac{1}{n+1} \left((s_1 - s_0) + 2(s_2 - s_1) + \ldots + n(s_n - s_{n-1}) \right) \\
&= \frac{1}{n+1} \left(n s_n - s_{n-1} - \ldots - s_1 - s_0 \right) \\
&= \frac{1}{n+1} ((n+1)s_n - s_n - s_{n-1} - \ldots - s_0) \\
&= s_n - \frac{s_n + s_{n-1} + \ldots + s_0}{n+1} \\
&= s_n - \sigma_n
\end{aligned}$$

Now, let $\varepsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} na_n = 0$, we can choose $N_1 \in \mathbb{N}$ such that for $n \geq N_1$ we have

$$|na_n| < \frac{\varepsilon}{2}$$

Let $c = \dfrac{1}{n+1}\displaystyle\sum_{k=1}^{N_1-1} ka_k$. We can choose N_2 such that $c < \frac{\varepsilon}{2}$ for $n \geq N_2$. Put $N = \max\{N_1, N_2\}$. It follows that for $n \geq N$

$$\begin{aligned}\left|\frac{1}{n+1}\sum_{k=1}^{n} ka_k\right| &\leq \left|\frac{1}{n+1}\sum_{k=1}^{N_1-1} ka_k\right| + \left|\frac{1}{n+1}\sum_{k=N_1}^{n} ka_k\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

Hence $\lim_{n\to\infty} s_n - \sigma_n = 0$. Because $\{\sigma_n\}$ converges, it follows that $\{s_n\}$ converges by the algebraic limit theorem.

□

(e) Derive the last conclusion from a weaker hypothesis: Assume $M < \infty$, $|na_n| \leq M$ for all n, and $\lim \sigma_n = \sigma$. Prove that $\lim s_n = \sigma$, by completing the following outline. *We fill out the outline now.*
Proof. If $m < n$, then

$$\begin{aligned}s_n - \sigma_n &= s_n - \frac{n-m}{n-m}\cdot\frac{s_0 + \ldots + s_n}{n+1} \\ &= s_n - \frac{1}{n-m}\left(\frac{n-m}{n+1}(s_0 + \ldots + s_n)\right) \\ &= s_n - \frac{1}{n-m}\left(\frac{(n+1)-(m+1)}{n+1}(s_0 + \ldots + s_n)\right) \\ &= s_n - \frac{1}{n-m}\left(-\frac{(m+1)}{n+1}(s_0 + \ldots + s_n) + \frac{m+1}{m+1}(s_0 + \ldots + s_n)\right) \\ &= s_n - \frac{1}{n-m}(s_{m+1} + \ldots + s_n) - \frac{1}{n-m}\left((m+1)\sigma_m - (m+1)\sigma_n\right) \\ &= s_n - \frac{1}{n-m}(s_{m+1} + \ldots + s_n) - \frac{m+1}{n-m}(\sigma_m - \sigma_n) \\ &= \frac{1}{n-m}(s_n - s_{m+1} + s_n - s_{m+2} + \ldots + s_n - s_n) + \frac{m+1}{n-m}(\sigma_n - \sigma_m) \\ &= \frac{1}{n-m}\sum_{i=m+1}^{n}(s_n - s_i) + \frac{m+1}{n-m}(\sigma_n - \sigma_m)\end{aligned}$$

For these i,

$$|s_n - s_i| = \left|\sum_{k=i+1}^{n} a_k\right| \leq \frac{(n-i)M}{i+1} \leq \frac{(n-m-1)M}{m+2}$$

The second inequality follows since $\frac{n-x}{x+1} = \frac{n+1}{x+1} - 1$, so as x increases, this quantity decreases. Hence for values of x between $m+1$ and n, the greatest value is at

$m+1$.
Fix $\varepsilon > 0$ and associate with each n the integer m that satisfies

$$m \leq \frac{n-\varepsilon}{1+\varepsilon} < m+1$$

Then $\frac{m+1}{n-m} \leq \frac{1}{\varepsilon}$ and $|s_n - s_i| < M\varepsilon$. Hence

$$\limsup_{n\to\infty} |s_n - \sigma| \leq M\varepsilon$$

Since ε was arbitrary, $\lim s_n = \sigma$. Note that the key idea here was choosing m, n in a particular way that yields the desired result.

□

15. Definition 3.21 can be extended to the case in which the a_n lie in some fixed $\mathbb{R}^k$. Absolute convergence is defined as convergence of $\sum |\mathbf{a_n}|$. Show that Theorem 3.22, 3.23, 3.25(a), 3.33, 3.34, 3.42, 3.45, 3.47, and 3.55 are true in this more general setting. (Only slight modifications are required in any of the proofs)
Omitted.

16. Fix a positive number α. Choose $x_1 > \sqrt{\alpha}$, and define $x_2, x_3, x_4, \ldots$ by the recursion formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right)$$

(a) Prove that $\{x_n\}$ decreases monotonically and that $\lim x_n = \sqrt{\alpha}$.
Proof. Since $x_1 > \sqrt{\alpha}$, it follows that

$$x_1 = \frac{x_1^2 + x_1^2}{2x_1} > \frac{1}{2}\left(x_1 + \frac{\alpha}{x_1}\right) = x_2$$

Also

$$x_1^2 - 2x_1\sqrt{\alpha} + \alpha = (x_1 - \sqrt{\alpha})^2 > 0$$

So by rearrangement

$$x_2 = \frac{1}{2}\left(x_1 + \frac{\alpha}{x_1}\right) > \sqrt{\alpha}$$

Now, for the inductive step we may assume $x_n > \sqrt{\alpha}$ to conclude that

$$x_n = \frac{x_n^2 + x_n^2}{2x_n} > \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) = x_{n+1}$$

Hence, $\{x_n\}$ decreases monotonically. Further, it is bounded since we see that $x_1 > x_n > \sqrt{\alpha}$ for all $n \in \mathbb{N}$. Thus, the sequence converges by Theorem 3.14 (MCT). Since the sequence converges, we know that $\lim x_{n+1} = \lim x_n$. If we let x be this limit, then it follows that

$$x = \frac{1}{2}\left(x + \frac{\alpha}{x}\right)$$

Solving for x shows $x = \sqrt{\alpha}$ as claimed.

□

(b) Put $\varepsilon_n = x_n - \sqrt{\alpha}$, and show that

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$$

so that, setting $\beta = 2\sqrt{\alpha}$,

$$\varepsilon_{n+1} < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n} \qquad (n = 1, 2, 3, \ldots)$$

Solution. Above, we have shown that $x_n > \sqrt{a}$ for all $n \in \mathbb{N}$. Hence, the claim $\frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{a}}$ follows immediately. For the second claim, notice that

$$\varepsilon_2 = \frac{\varepsilon_1^2}{2x_1} \quad \varepsilon_3 = \frac{\varepsilon_2^2}{2x_2} = \frac{1}{2x_2}\left(\frac{\varepsilon_1^2}{2x_1}\right)^2 = \frac{\varepsilon_1^4}{2x_2 \cdot 4x_1^2}$$

Continuing the pattern, we see that

$$\varepsilon_{n+1} = \frac{(\varepsilon_1)^{2^n}}{\prod_{i=0}^{n-1} 2^{2^i} x_{n-i}^{(2^{2^i})}} < \frac{(\varepsilon_1)^{2^n}}{2^{1+2+\ldots+2^n} \cdot x_n^{2^n-1}} < \frac{(\varepsilon_1)^{2^n}}{2^{2^n-1}(\sqrt{a})^{2^n-1}}$$

from which the desired inequality follows.

(c) This is a good algorithm for computing square roots, since the recursion formula is simple and the convergence is extremely rapid. For example, if $\alpha = 3$ and $x_1 = 2$, show that $\frac{\varepsilon_1}{\beta} < \frac{1}{10}$ and that therefore

$$\varepsilon_5 < 4 \cdot 10^{-16}, \qquad \varepsilon_6 < 4 \cdot 10^{-32}$$

Solution. Observe that

$$10 - 6\sqrt{3} < 0$$

Hence, we may directly compute that

$$\frac{\varepsilon_1}{\beta} = \frac{2-\sqrt{3}}{2\sqrt{3}} = \frac{20-10\sqrt{3}}{20\sqrt{3}} < \frac{2\sqrt{3}}{20\sqrt{3}}$$

We also have

$$\varepsilon_5 < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^4} < 2\sqrt{3}\left(\frac{1}{10}\right)^{16} < 4 \cdot 10^{-16}$$

and a similar relation holds for ε_6.

□

17. Fix $\alpha > 1$. Take $x_1 > \sqrt{\alpha}$, and define

$$x_{n+1} = \frac{\alpha + x_n}{1 + x_n} = x_n + \frac{\alpha - x_n^2}{1 + x_n}$$

(a) Prove that $x_1 > x_3 > x_5 > \cdots$.

First, we remark that

$$x_n + \frac{\alpha - x_n^2}{1 + x_n} = \frac{x_n + x_n^2 + \alpha - x_n^2}{1 + x_n} = \frac{\alpha + x_n}{1 + x_n}$$

Suppose that n is odd and $x_n > \sqrt{\alpha}$. We have

$$\begin{aligned} x_{n+2} &= x_{n+1} + \frac{\alpha - x_{n+1}^2}{1 + x_{n+1}} \\ &= x_n + \frac{\alpha - x_n^2}{1 + x_n} + \frac{\alpha - x_{n+1}^2}{1 + x_{n+1}} \end{aligned}$$

By calculating common denominators, we see that

$$x_{n+2} = x_n + \frac{(x_n^2 - \alpha)(-2x_n - 2)}{(1 + x_n)^2 + (\alpha + x_n)(1 + x_n)} < x_n$$

Now, we want to show that $x_{n+2} > \sqrt{\alpha}$. We claim that $x_n^2 - x_{n+2}^2 < x_n^2 - \alpha$. Notice that

$$\begin{aligned} x_{n+2}^2 &= x_n^2 + x_n \cdot \frac{(x_n^2 - \alpha)(-2x_n - 2)}{(1 + x_n)^2 + (\alpha + x_n)(1 + x_n)} + \left(\frac{(x_n^2 - \alpha)(-2x_n - 2)}{(1 + x_n)^2 + (\alpha + x_n)(1 + x_n)}\right)^2 \\ &= x_n^2 - \frac{(x_n^2 - \alpha)(2x_n^2 + 2x_n)}{(1 + x_n)^2 + (\alpha + x_n)(1 + x_n)} + \left(\frac{(x_n^2 - \alpha)(-2x_n - 2)}{(1 + x_n)^2 + (\alpha + x_n)(1 + x_n)}\right)^2 \end{aligned}$$

Then

$$\begin{aligned} x_n^2 - x_{n+2}^2 &= \frac{(x_n^2 - \alpha)(2x_n^2 + 2x_n)}{(1 + x_n)^2 + (\alpha + x_n)(1 + x_n)} - \left(\frac{(x_n^2 - \alpha)(-2x_n - 2)}{(1 + x_n)^2 + (\alpha + x_n)(1 + x_n)}\right)^2 \\ &< \frac{(x_n^2 - \alpha)(2x_n^2 + 2x_n)}{(1 + x_n)^2 + (\alpha + x_n)(1 + x_n)} \\ &< x_n^2 - \alpha \end{aligned}$$

where the last inequality follows since

$$\frac{(2x_n^2 + 2x_n)}{(1 + x_n)^2 + (\alpha + x_n)(1 + x_n)} < 1$$

$\square$

(b) Prove that $x_2 < x_4 < x_6 \cdots$.

Assume that n is even and assume that $x_{n-1} > \sqrt{\alpha}$. By definition

$$x_n = x_{n-1} + \frac{\alpha - x_{n-1}^2}{1 + x_{n-1}}$$

Since $x_{n-1} > \sqrt{\alpha}$, it follows that $\frac{\alpha - x_{n-1}^2}{1 + x_{n-1}}$ is negative and $x_n < x_{n-1}$. Hence

$$0 < x_n - x_{n-1} = \frac{\alpha - x_{n-1}^2}{1 + x_{n-1}}$$

By rearrangement, we see that

$$\alpha > x_{n-1}^2 > x_n^2$$

Since $x_n^2 - \alpha < 0$, we conclude that

$$x_{n+2} = x_n + \frac{(x_n^2 - \alpha)(-2x_n - 2)}{(1 + x_n)^2 + (\alpha + x_n)(1 + x_n)} > x_n$$

as desired.

□

(c) Prove that $\lim x_n = \sqrt{\alpha}$.
Proof. Consider the sequence $a_n = |x_n - \sqrt{\alpha}|$. Let us consider the two subsequences of odd and even indices. We have shown that $x_1 > x_3 > \cdots$ and $x_n > \sqrt{\alpha}$ for all odd n. This shows that the subsequence is monotonic and bounded and so it converges. Call x the limit. We see that

$$\begin{aligned} x &= \frac{\alpha + x}{1 + x} \\ x^2 + x - x &= \alpha \\ x &= \sqrt{\alpha} \end{aligned}$$

A similar argument shows that the subsequence with even indices converges to $\sqrt{\alpha}$. Since both even and odd indexed subsequences converge to $\sqrt{\alpha}$, it follows that $a_n \to 0$. Hence $x_n \to \sqrt{\alpha}$.

□

(d) Compare the rapidity of convergence of this process with the one described in Exercise 16.
For the odd indices, since $x_n > \sqrt{\alpha}$, we can make a similar argument to that in 3.16(c) regarding the rapidity of the convergence. So we see that this sequence in the current exercise converges at least twice as slowly.

18. Replace the recursion formula of Exercise 16 by

$$x_{n+1} = \frac{p-1}{p} x_n + \frac{\alpha}{p} x_n^{-p+1}$$

where p is a fixed positive integer, and describe the behavior of the resulting sequences $\{x_n\}$.
Proof. Choose $x_1 > \alpha^{\frac{1}{p}}$. Assume $x_n > \alpha^{\frac{1}{p}}$. Then we have

$$x_n = \frac{x_n^p + x_n^p + \ldots + x_n^p}{p x_n^{p-1}} > \frac{p-1}{p} x_n + \frac{\alpha}{p} x_n^{-p+1} = x_{n+1}$$

which shows that the sequence is monotonic. Further, notice that x_{n+1} is the sum and product of positive real numbers, hence, it must be positive as well. So, $\{x_n\}$ is bounded between x_1 and 0. Hence, it converges by MCT. Let the limit it converges to be x. Then we have

$$\begin{aligned} x &= \frac{p-1}{p}x + \frac{\alpha}{p}x^{-p+1} \\ px - (p-1)x &= \alpha x^{-p+1} \\ x^p &= \alpha \end{aligned}$$

This shows that $\lim x_n = \alpha^{\frac{1}{p}}$.

$\square$

19. Associate to each sequence $a = \{\alpha_n\}$, in which α_n is 0 or 2, the real number

$$x(a) = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}$$

Prove that the set of all $x(a)$ is precisely the Cantor set described in Section 2.44.
Proof. Let S be the set of all $x(a)$ and consider an $x(a) \in S$. Call the Cantor set K. Then we may write

$$x(a) = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}$$

Now, by construction, the endpoints of all of the intervals in Cantor set are contained in the Cantor set. We construct a sequence $\{x_n\} \subseteq K$ inductively. We start with the interval $I_0 = [0, 1]$. Suppose we have selected one of the intervals that remain after the $(n-1)^{st}$ step, I_{n-1}. Let L and R be the intervals after the removal of the middle third from I_{n-1} on the n^{th} iteration. If $\alpha_n = 0$, select x_n as the left endpoint of L and set $I_n = L$. If $\alpha_n = 2$, select x_n as the left endpoint of R and set $I_n = R$.
We claim that

$$x_n = \sum_{k=1}^{n} \frac{\alpha_k}{3^k}$$

Begin with $I_0 = [0, 1]$. It is clear that if $\alpha_1 = 0$, then $x_1 = 0$ and if $\alpha_1 = 2$, then $x_1 = \frac{2}{3}$, so the claim holds. Notice that the choice of I_1 corresponds to $[x_1, x_1 + \frac{1}{3}]$. Now, suppose that for some $m \in \mathbb{N}$

$$x_m = \sum_{k=1}^{m} \frac{\alpha_k}{3^k}$$

The interval I_m is given by $[x_m, x_m + \frac{1}{3^{m-1}}]$. After removing the middle third, we have

$$L = \left[x_m, x_m + \frac{1}{3^m}\right], \quad R = \left[x_m + \frac{2}{3^m}, x_m + \frac{1}{3^{m-1}}\right]$$

If $\alpha_{m+1} = 0$, then clearly

$$x_{m+1} = \sum_{k=1}^{m+1} \frac{\alpha_k}{3^k} = \sum_{k=1}^{m} \frac{\alpha_k}{3^k} = x_m$$

This shows that choosing the left endpoint of L is the same as adding the $m+1$ term in the series. If $a_{m+1} = 2$, then

$$x_{m+1} = \sum_{k=1}^{m+1} \frac{\alpha_k}{3^k} = x_m + \frac{2}{3^m}$$

which corresponds to the left endpoint of R. Therefore, x_n is exactly the sequence of partial sums of $x(a)$ meaning that they must converge to the same point. Hence $x(a) \in K$ since it is closed. The reverse inclusion is similar.

$\square$

20. Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X, and some subsequence $\{p_{n_l}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p.
Proof. Let $\varepsilon > 0$ be arbitrary. Since $p_{n_l} \to p$, there exists some $N_1 \in \mathbb{N}$ such that for all $l \geq N_1$,
$$|p_{n_l} - p| < \frac{\varepsilon}{2}$$
Since $\{p_n\}$ is Cauchy, there exists some $N_2 \in \mathbb{N}$ such that for all $n, m \geq N_2$,
$$|p_n - p_m| < \frac{\varepsilon}{2}$$
Choose $N = \max\{N_1, N_2\}$. It follows that for $l \geq N$ we have
$$|p_l - p| \leq |p_l - p_{n_l}| + |p_{n_l} - p| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$
Hence, p_n converges to p.

$\square$

21. Prove the following analogue of Theorem 3.10(b): If $\{E_n\}$ is a sequence of closed, nonempty, and bounded sets in a *complete* metric space X, if $E_n \supseteq E_{n+1}$ and if
$$\lim \operatorname{diam} E_n = 0,$$
then $\bigcap_{n=1}^{\infty} E_n$ consists exactly of one point.
Proof. Construct a sequence $\{x_n\}$ by choosing $x_n \in E_n$ for each $n \in \mathbb{N}$. This choice is possible as each E_n is nonempty. The boundedness condition gives us finite diameters. Since $E_n \supseteq E_{n+1}$ for all $n \in \mathbb{N}$, it follows that E_N contains $x_N, x_{N+1}, x_{N+2}, \ldots$. Fix an $\varepsilon > 0$. Since $\lim \operatorname{diam} E_n = 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ means $\operatorname{diam} E_n < \varepsilon$. But then for $n, m \geq N$, we have
$$|x_n - x_m| < \operatorname{diam} E_N < \varepsilon$$
Hence $\{x_n\}$ is Cauchy. Because X is a complete metric space, which means that every Cauchy sequence converges, there exists some $x \in X$ satisfying $x_n \to x$. Further, x is a limit point for each E_n. Since every E_n is closed, it follows that $x \in E_n$ for all $n \in \mathbb{N}$, hence $x \in \bigcap_{n=1}^{\infty} E_n$. This point must be unique, otherwise $\lim \operatorname{diam} E_n > 0$.

$\square$

22. Suppose X is a nonempty complete metric space, and $\{G_n\}$ is a sequence of dense open subsets of X. Prove Baire's theorem, namely, that $\bigcap_{n=1}^{\infty} G_n$ is not empty. (In fact, it is dense in X). *Hint:* Find a shrinking sequence of neighborhoods E_n such that $\overline{E_n} \subseteq G_n$ and apply Exercise 21.

Proof. Let $x \in X$ and $\varepsilon > 0$ be given. Since G_1 is dense in X, x is a limit point of G_1. Hence, there exists $g_1 \in G_1 \cap N_\varepsilon(x)$. Since $G_1 \cap N_\varepsilon(x)$ is open, there exists $\delta_1 > 0$ such that $\overline{N_{\delta_1}(g_1)} \subseteq G_1 \cap N_\varepsilon(x)$. Since G_2 is dense in X, there exists $g_2 \in G_2 \cap N_{\delta_1}(g_1)$. Again, since the intersection of open sets is open, there exists a $\delta_2 > 0$ such that $\overline{N_{\delta_2}(g_2)} \subseteq G_2 \cap N_{\delta_1}(g_1)$. Continuing this process, we find a shrinking sequence of neighborhoods whose closures are contained in each G_n. Additionally, at each step, if $\frac{1}{2}\delta_k \leq \delta_{k+1}$, then choose δ_{k+1} such that $\frac{1}{2}\delta_k > \delta_{k+1}$ instead. This choice is valid the closed set with a smaller radius will still be contained in the intersecting open sets. Setting

$$E_n = \overline{N_{\delta_n}(g_n)}$$

we have a sequence of closed, nonempty, and bounded sets in a complete metric space such that $E_n \supseteq E_{n+1}$ and $\lim \operatorname{diam} E_n = 0$. Hence $\bigcap_{n=1}^{\infty} E_n$ is nonempty by Exercise 21.

□

23. Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges.

Proof. For any $m, n \in \mathbb{N}$, we have

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n)$$

Fix an $\varepsilon > 0$. Since $\{p_n\}$ and $\{q_n\}$ are Cauchy, there exist $N_1 \in \mathbb{N}$ such that $d(p_n, p_m) < \frac{\varepsilon}{2}$ for all $n, m \geq N_1$ and $N_2 \in \mathbb{N}$ such that $d(q_n, q_m) < \frac{\varepsilon}{2}$. Hence, taking $N = \max\{N_1, N_2\}$, we have

$$|d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_m, q_n) < \varepsilon$$

for all $n, m \geq N$. Therefore, $\{d(p_n, q_n)\}$ converges.

□

24. Let X be a metric space.

(a) Call two Cauchy sequences $\{p_n\}, \{q_n\}$ in X equivalent if

$$\lim_{n\to\infty} d(p_n, q_n) = 0.$$

Prove that this is an equivalence relation.

We verify the properties of an equivalence relation. We will denote the equivalence of two sequences using $\sim$.

Reflexivity. Let $(p_n)_{n=1}^{\infty}$ be any Cauchy sequence in X. From the definition of a metric space, we know that $d(p_n, p_n) = 0$ for all n, since $p_n = p_n$. The limit of the constant sequence of zeroes is clearly zero.

Symmetry. Suppose $(p_n)_{n=1}^{\infty}$ and $(q_n)_{n=1}^{\infty}$ are two Cauchy sequences that are equivalent. Notice that

$$d(p_n, q_n) = d(q_n, p_n) \text{ for all } n \in \mathbb{N}$$

As a result, we know that:

$$\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(q_n, p_n) = 0$$

Transitivity. Suppose $(p_n)_{n=1}^{\infty}$, $(q_n)_{n=1}^{\infty}$, and $(r_n)_{n=1}^{\infty}$ are three Cauchy sequences such that $(p_n) \sim (q_n)$ and $(q_n) \sim (r_n)$. Then, by the definition of a metric space, we have:

$$d(p_n, r_n) \leq d(p_n, q_n) + d(q_n, r_n) \text{ for all } n \in \mathbb{N}$$

By the Algebraic Limit Theorem, we know that

$$\lim_{n\to\infty} d(p_n, q_n) + d(q_n, r_n) = 0$$

Since $d(p_n, r_n) \geq 0$ for all $n \in \mathbb{N}$, by the Squeeze Theorem, we see that $\lim_{n\to\infty} d(p_n, r_n) = 0$, so $(p_n) \sim (r_n)$ as desired.

(b) Let X^* be the set of all equivalence classes so obtained. If $P \in X^*$, $Q \in X^*$, $\{p_n\} \in P$, $\{q_n\} \in Q$, define

$$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n)$$

by Exercise 23, this limit exists. Show that the number $\Delta(P, Q)$ is unchanged if $\{p_n\}$ and $\{q_n\}$ are replaced by equivalent sequences, and hence that Δ is a distance function in X^*.
Suppose $(p_n) \sim (p_n')$ and $(q_n) \sim (q_n)'$. Then it follows that:

$$\lim_{n\to\infty} d(p_n', q_n') \leq \lim_{n\to\infty} [d(p_n', p_n) + d(p_n, q_n) + d(q_n, q_n')] = \lim_{n\to\infty} d(p_n, q_n)$$

Further, we can exchange the positions of the sequences due to the symmetry of the metric space. Therefore we conclude that $\lim_{n\to\infty} d(p_n', q_n') = \lim_{n\to\infty} d(p_n, q_n)$.
It is clear from the definition of Δ that $\Delta(P, Q) \geq 0$ for all $P, Q \in X^*$.

(1) We verify $\Delta(P, Q) = 0$ if and only if $P = Q$. The forward direction follows directly from the definition of equivalence in part (a). That is, if $\lim_{n\to\infty} d(p_n, q_n) = 0$, then $(p_n) \sim (q_n)$. The reverse direction follows from what we just proved above.
(2) The equality $\Delta(P, Q) = \Delta(Q, P)$ follows directly from the definition of Δ.
(3) Given $P, Q, R \in X^*$, choose $\{p_n\} \in P, \{q_n\} \in Q$, and $\{r_n\} \in R$. Then by definition and by the triangle inequality,

$$\begin{aligned} \Delta(P, Q) &= \lim_{n\to\infty} d(p_n, q_n) \\ &\leq \lim_{n\to\infty} [d(p_n, r_n) + d(r_n, q_n)] \\ &= \lim_{n\to\infty} d(p_n, r_n) + \lim_{n\to\infty} d(r_n, q_n) \\ &= \Delta(P, R) + \Delta(R, Q) \end{aligned}$$

Therefore, we conclude the Δ is a metric in X^*.

(c) Prove that the resulting metric space X^* is complete.
Suppose $\{P_n\}$ is a Cauchy sequence in X^*, and choose $\{p_k^n\} \in P_n$ for $n = 1, 2, \ldots$. For each n, there exists a positive integer K_n such that $k, l > K_n$ implies $d(p_k^n, p_l^n) < 2^{-n}$. Define $p_n = p_{K_n}^n$ for $n \in \mathbb{N}$. We show that $\{p_n\}$ is Cauchy in X.
Given $\varepsilon > 0$, since $\{P_n\}$ is Cauchy in X^*, there exists a positive integer N such that $m, n \geq N$ implies $\Delta(P_m, P_n) < \varepsilon$, or

$$\lim_{k\to\infty} d(p_k^m, p_k^n) < \varepsilon$$

Let N' be a positive integer such that $2^{-N'} < \varepsilon$. Then, $m, n \geq \max\{N, N'\}$ implies

$$\begin{aligned} d(p_m, p_n) &= d(p_{K_m}^m, p_{K_n}^n) \\ &\leq d(p_{K_m}^m, p_k^m) + d(p_k^m, p_k^n) + d(p_k^n, p_{K_n}^n) \\ &< 2^{-m} + \varepsilon + 2^{-n} < 3\varepsilon \end{aligned}$$

where the integer k can be chosen sufficiently large such that $k > \max\{K_m, K_n\}$ and $d(p_k^m, p_k^n) < \varepsilon$. Hence, $\{p_n\}$ is Cauchy in X.
Let $P \in X^*$ be such that $\{p_n\} \in P$. It suffices to show that $P_n \to P$, which completes the proof. Let $\varepsilon > 0$ be arbitrary. Put $n \geq \max\{N, N'\}$, $k \geq \max\{N, N', K_n\}$, and $m \geq \max\{K_n, K_k\}$ sufficiently large such that $d(p_m^n, p_m^k) < \varepsilon$. We have

$$\begin{aligned} d(p_k^n, p_k) &= d(p_k^n, p_{K_k}^k) \\ &\leq \underbrace{d(p_k^n, p_m^n)}_{p^n \text{ is Cauchy and } n,k\geq N} + \underbrace{d(p_m^n, p_m^k)}_{\{P_n\} \text{ is Cauchy, using limit}} + \underbrace{d(p_m^k, p_{K_k}^k)}_{p^k \text{ is Cauchy and } m,K_k\geq K_k} \\ &< 2^{-n} + \varepsilon + 2^{-k} < 3\varepsilon \end{aligned}$$

which follows that $\lim_{k\to\infty} d(p_k^n, p_k) \leq 3\varepsilon$ or $\Delta(P_n, P) \leq 3\varepsilon$. Hence $P_n \to P$.

□

Observe that taking $p_n = p_n^n$, the diagonal sequence, does not work. Let $(p_k^n)_{k\in\mathbb{N}}$ be the sequence of n ones followed by an infinite tail of zeroes. Then $p_n = p_n^n$ yields a constant sequence of ones, but every sequence (p_k^n) belongs to the same equivalence class of sequences converging to 0.

(d) For each $p \in X$, there is a Cauchy sequence all of whose terms are p; let P_p be the element of X^* which contains this sequence. Prove that

$$\Delta(P_p, P_q) = d(p, q)$$

for all $p, q \in X$. In other words, the mapping φ defined by $\varphi(p) = P_p$ is an isometry of X into X^*.
Notice that if $p_n = p$ and $q_n = q$ for all n, then it follows that

$$\lim_{n\to\infty} d(p_n, q_n) = d(p, q)$$

(e) Prove that $\varphi(X)$ is dense in X^*, and that $\varphi(X) = X^*$ if X is complete. By (d), we may identify X and $\varphi(X)$ and thus regard X as embedded in the complete metric space X^*. We call X^* the completion of X.

(i) Let $P \in X^*$ and $\{p_n\} \in P$, and let $\varepsilon > 0$ be arbitrary. Since $\{p_n\}$ is Cauchy, there exists a positive integer N such that $m, n \geq N$ implies $d(p_m, p_n) < \varepsilon$. It follows that

$$\Delta(P, \varphi(p_N)) = \Delta(P, P_{p_N}) = \lim_{n \to \infty} d(p_n, p_N) \leq \varepsilon < 2\varepsilon$$

That is, the neighborhood of P with radius 2ε contains an element $\varphi(p_N) \in \varphi(X)$. So we conclude that $\varphi(X)$ is dense in X^*. In our language of the definition of dense in lecture, we have shown that $B(x, r) \cap E \neq \varnothing$ for all $x \in X$ and $r > 0$, where X stands for X^* and r is analogous to the arbitrary choice of ε.

(ii) If X is complete, let $P \in X^*$ and $\{p_n\} \in P$. since $\{p_n\}$ is Cauchy, $p_n \to p$ for some $p \in X$, i.e., we have

$$\Delta(P, P_p) = \lim_{n \to \infty} d(p_n, p) = 0$$

So $P = P_p$, and hence $X^* = \varphi(X)$.

25. Let X be the metric space whose points are the rational numbers, with the metric $d(x, y) = |x - y|$. What is the completion of this space?

By Theorem 3.11(c) in Rudin, every rational Cauchy sequence converges to a real number. Additionally, for every $x \in \mathbb{R}$, there exists a rational Cauchy sequence $\{p_n\}$ such that $\lim_{n \to \infty} p_n = x$. Thus, the completion of X is $\varphi(\mathbb{R})$.

$\square$

Chapter 4: Continuity

1. Suppose f is a real function defined on $\mathbb{R}^1$ which satisfies

$$\lim_{h\to 0}[f(x+h)-f(x-h)]=0$$

for every $x \in \mathbb{R}^1$. Does this imply that f is continuous?
Solution. Consider the function

$$f(x)=\begin{cases} x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

If $f(x)$ is continuous at x, then we know that

$$\lim_{h\to 0} f(x+h)=\lim_{h\to 0} f(x-h)=f(x)$$

Since f is continuous everywhere except $x=0$, the above limit is satisfied for all $\mathbb{R}\setminus\{0\}$. At $x=0$, this limit is also satisfied, since

$$|f(0+h)-f(0-h)|=|2h|$$

Hence, we may simply choose $\delta < \varepsilon$ to see that $|f(h)-f(-h)|<\varepsilon$. Thus, this limit is satisfied for all $x \in \mathbb{R}$. However, f is not continuous at $x=0$, as $f(0)=1\neq 0=\lim_{x\to p} f(x)$.

□

2. If f is a continuous mapping of a metric space X into a metric space Y, prove that

$$f(\overline{E}) \subseteq \overline{f(E)}$$

for every set $E \subseteq X$. ($\overline{E}$ denotes the closure of E.) Show, by an example, that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$.
Proof. Let $y \in f(\overline{E})$. By definition, there is some $x \in \overline{E}$ such that $f(x)=y$. If $x \in E$, then $f(x) \in f(E) \subseteq \overline{f(E)}$ and the relation holds. If $x \notin E$, then it must be a limit point of E. Hence, by Theorem 3.2(d), there is a sequence $\{x_n\} \subseteq E$ such that $\lim_{n\to\infty} x_n = x$. Since f is continuous, by Theorem 4.6 we know that $\lim_{x'\to x} f(x')=f(x)$. From Theorem 4.2, we know that $\lim_{x'\to x} f(x')=f(x)$ if and only if

$$\lim_{n\to\infty} f(p_n)=f(x)$$

for every sequence $\{p_n\} \subseteq E$ such that $p_n \neq x$ and $\lim_{n\to\infty} p_n = x$. The sequence $\{x_n\}$ satisfies these conditions, hence

$$\lim_{n\to\infty} f(x_n)=y$$

This shows that y is a limit point of $f(E)$. Hence $y \in \overline{f(E)}$ as claimed.

□

For an example, consider $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}$ given by $f(x)=x$. Then $f(\overline{E})=\mathbb{R}\setminus\{0\}$ but $\overline{f(E)}=\mathbb{R}$. Another example is $f(x)=\arctan(x)$ over the reals.

3. Let f be a continuous real function on a metric space X. Let $Z(f)$ (the *zero set* of f). be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.
Proof. According to the above, $Z(f)$ is the preimage of the set $\{0\}$. The set $\{0\}$ is closed because it is finite and so it has no limit points. Since f is continuous, the preimages of closed sets are closed by the Corollary to Theorem 4.8. Hence $Z(f)$ is closed.

$\square$

4. Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(p)$ for all $p \in E$, prove that $g(p) = f(p)$ for all $p \in X$. (In other words, a continuous mapping is determined by its values on a dense subset of its domain.)
Proof. Let $x \in X$ be arbitrary. Since E is a dense subset of X, it follows that x is a limit point of E or a point of E. If $x \in E$, then $f(x) \in f(E)$. If x is a limit point of E and $x \notin E$, then there exists a sequence $\{x_n\} \subseteq E$ such that $x_n \to x$. By continuity, we know that
$$\lim_{n\to\infty} f(x_n) = f(x)$$
Since $\{f(x_n)\} \subseteq f(E)$, it follows that $f(x)$ is a limit point of $f(E)$. Thus, $f(x) \in f(E)$ or $f(x)$ is a limit point of $f(E)$, so $f(E)$ is dense in $f(X)$.
Now, suppose that $g(p) = f(p)$ for all $p \in E$. Choose an $x \in X$. Since g and f are continuous, for all arbitrary sequences $\{x_n\} \subseteq E$ with $x_n \neq x$ and $x_n \to x$, we have
$$g(x) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} f(x_n) = f(x)$$
Since x is a limit point of E, there exists such a sequence $\{x_n\} \subseteq E$ for which the above conditions hold.

$\square$

5. If f is a real continuous function defined on a closed set $E \subseteq \mathbb{R}^1$, prove that there exist continuous real functions g on $\mathbb{R}^1$ such that $g(x) = f(x)$ for all $x \in E$. (Such functions g are called *continuous extensions* of f from E to $\mathbb{R}^1$.) Show that the result becomes false if the word "closed" is omitted. Extend the result to vector-valued functions. *Hint:* Let the graph of g be a straight line on each of the segments which constitute the complement of E (compare Exercise 29, Chap. 2). The result remains true if $\mathbb{R}^1$ is replaced by any metric space, but the proof is not so simple.
Proof. Since E is closed, we know that E^c is open. By Exercise 2.29, it follows that E^c is the disjoint union of open segments. Let $a, b \in \mathbb{R}$. If $(a, \infty) \subseteq E^c$ or $(-\infty, b) \subseteq E^c$, we can define $g(x) = f(a)$ for all $x \in (a, \infty)$ in the former case, and $g(x) = f(b)$ for all $x \in (-\infty, b)$ in the latter case. If $(a, b) \subseteq E^c$, define g on (a, b) by
$$g(x) = \left(\frac{x-a}{b-a}\right)(f(b) - f(a)) + f(a)$$
which is geometrically a straight line from $f(a)$ to $f(b)$ over the segment (a, b). The function g is a continuous extension of f.
To show that the result is false when closed is omitted, consider that $f(x) = \frac{1}{x}$ has

no continuous extension. To extend the result to vector-valued functions, consider a function $\mathbf{f} : \mathbb{R} \to \mathbb{R}^n$ given by $x \mapsto (f_1(x), \ldots, f_n(x))$. From the first proof we gave, there exists continuous extensions $g_1, \ldots, g_n$ for $f_1, \ldots, f_n$ respectively. Then $x \mapsto (g_1(x), \ldots, g_n(x))$ defines a continuous extension from $\mathbf{f}$ to $\mathbf{g}$.

$\square$

6. If f is defined on E, the *graph* of f is the set of points $(x, f(x))$, for $x \in E$. In particular, if E is a set of real numbers, and f is real-valued, the graph of f is a subset of the plane.
   Suppose E is compact, and prove that f is continuous on E if and only if its graph is compact.
   Proof. Let f be continuous. Consider another function $g : X \to X^2$ defined by
   $$g(x) = (x, f(x))$$
   We know g is continuous since each of its components is continuous (Theorem 4.10). By Theorem 4.14, since g is continuous and E is compact, $g(E)$, which is the graph, must be compact.
   On the other hand, suppose that the graph of f, denoted F, is compact. Consider $x \in E$. If x is an isolated point, then f is continuous at x. Assume that x is a limit point of E. Since E is a compact subset of a metric space, by Theorem 2.34, it is closed. Since E is closed, $x \in E$. Hence, $(x, f(x)) \in F$. Consider a sequence $\{x_n\} \subset E$ satisfying $x_n \to x$ with $x_n \neq x$. Since $\{(x_n, f(x_n))\}$ is a sequence in the compact metric space F, it contains a subsequence $\{(x_{n_k}, f(x_{n_k}))\}$ which converges to some point $(x', y') \in F$. However, since $x_n \to x$, it follows that $x_{n_k} \to x$. Hence, $x' = x$ and $y' = f(x)$, since F is the graph of f. As this applies to every subsequence x_{n_k}, it follows that $f(x_n) \to f(x)$. Thus, f is continuous at x. Together, these arguments show that f is continuous on E.

   $\square$

7. If $E \subseteq X$ and if f is a function defined on X, the *restriction* of f to E is the function g whose domain of definition is E, such that $g(p) = f(p)$ for $p \in E$. Define f and g on $\mathbb{R}^2$ by: $f(0,0) = g(0,0) = 0$, $f(x,y) = \frac{xy^2}{x^2+y^4}$, $g(x,y) = \frac{xy^2}{x^2+y^6}$ if $(x,y) \neq (0,0)$. Prove that f is bounded on $\mathbb{R}^2$, that g is unbounded in every neighborhood of $(0,0)$, and that f is not continuous at $(0,0)$; nevertheless, the restrictions of both f and g to every straight line in $\mathbb{R}^2$ are continuous!
   Proof.

   (i) To show that f is bounded on $\mathbb{R}^2$, consider that
   $$0 \leq (x - y^2)^2 = x^2 - 2xy^2 + y^4$$
   Then, by rearrangement, we have
   $$\frac{xy^2}{x^2 + y^4} \leq \frac{1}{2}$$
   Hence, $|f(x,y)| \leq \frac{1}{2}$ for all $(x,y) \in \mathbb{R}^2$, and the conclusion follows.

(ii) Next, consider the values of g along the function (y^3, y). We see that

$$g(y^3, y) = \frac{y^3}{2y^6} = \frac{1}{2y^3}$$

Thus, $g(y^3, y) \xrightarrow[y\to 0]{} \infty$, showing that it is unbounded in every neighborhood around the origin.

(iii) Suppose, for contradiction, that f is continuous. Consider $h(t) = (t^2, t)$. This function is continuous since its component functions are continuous. Hence, $f \circ h$ must be continuous as it is the composition of continuous functions. However, $(f \circ h)(0) = 0$ and

$$(f \circ h)(t) = \frac{t^4}{t^4 + t^4} = \frac{1}{2}, \quad t \neq 0$$

This shows that $\lim_{t\to 0}(f \circ h)(t) = \frac{1}{2}$. Since this is not equal to $(f \circ h)(0)$, by Theorem 4.6 we see that $f \circ h$ is not continuous. ($\lightning$)

(iv) Since f and g are continuous at any point other than the origin, we only need to check the continuity of the restricted graphs to lines at the origin. We consider two cases.

Suppose $x = 0$. Then we have

$$f(0, y) = \frac{0}{y^4} = 0 \quad g(0, y) = \frac{0}{y^6} = 0$$

On the other hand, we can have the restriction to the line $y = mx$, which gives us

$$f(x, mx) = \frac{m^2x^3}{x^2 + m^4x^4} = \frac{m^2x}{1 + m^4x^2}$$

Hence $\lim_{x\to 0} f(x, mx) = 0$, so the restriction of f to a straight line is continuous. The argument for g follows a similar pattern.

□

8. Let f be a real uniformly continuous function on the bounded set E in $\mathbb{R}^1$. Prove that f is bounded on E.

Proof. Since f is uniformly continuous, then there exists $\delta > 0$ such that

$$|f(x_1) - f(x_2)| < 1$$

for all $x_1, x_2 \in E$ satisfying $|x_1 - x_2| < \delta$. Since E is bounded, it is possible to cover it with finitely many open intervals of length δ. Call these intervals $I_1, \dots, I_n$. We may choose an $x_k \in I_k$ for each $k \in \{1, \dots, n\}$. It follows that for all $x \in I_k$, we have $|f(x) - f(x_k)| < 1$. Call

$$M = \max\{f(x_1) + 1, \dots, f(x_n) + 1\}$$

We see that

$$|f(x)| < M$$

for all $x \in E$. To be explicit, we must have $x \in I_k$ because $I_1, \ldots, I_n$ cover E. But then $|f(x) - f(x_k)| < 1$, so $f(x) < f(x_k) + 1 \leq M$. Therefore, f is bounded on E.

$\square$

Show that the conclusion is false if boundedness of E is omitted from the hypothesis. Take $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$. This function is uniformly continuous as we may choose $\delta = \varepsilon$. However, $f(\mathbb{R}) = \mathbb{R}$, showing that the image of the function is not bounded.

9. Show that the requirement in the definition of uniform continuity can be rephrased as follows, in terms of diameters of sets: To every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\operatorname{diam} f(E) < \varepsilon$ for all $E \subseteq X$ with $\operatorname{diam} E < \delta$.
Solution. Suppose Definition 4.18 holds. Let $\varepsilon > 0$ be arbitrary. Since f is uniformly continuous, there exists a $\delta > 0$ such that

$$d(f(x), f(y)) < \frac{\varepsilon}{2}$$

for all $x, y \in X$ satisfying $d(x, y) < \delta$. Let $E \subseteq X$ satisfy $\operatorname{diam} E < \delta$. It follows that for all $p, q \in E$, we have

$$d(p, q) \leq \operatorname{diam} E < \delta$$

Hence, $d(f(p), f(q)) < \frac{\varepsilon}{2}$. This shows that $\operatorname{diam} f(E) \leq \frac{\varepsilon}{2} < \varepsilon$. So this alternative definition also holds. The reverse direction assuming first that the alternative definition holds is shown similarly.

10. Complete the details of the following alternative proof of Theorem 4.19: If f is not uniformly continuous, then for some $\varepsilon > 0$ there are sequences $\{p_n\}, \{q_n\}$ in X such that $d_X(p_n, q_n) \to 0$ but $d_Y(f(p_n), f(q_n)) > \varepsilon$. Use Theorem 2.37 to obtain a contradiction.
Proof. To be explicit, suppose for contradiction that f is a continuous mapping from a compact metric space X to a metric space Y and that f is not uniformly continuous on X. By Theorem 3.6 (which relies on Theorem 2.37), we know that there are subsequences $\{p_{n_k}\}$ and $\{q_{n_k}\}$ which converge to some point $x \in X$. Since f is continuous at x, we know that

$$\lim_{k\to\infty} f(p_{n_k}) = \lim_{k\to\infty} f(q_{n_k}) = f(x)$$

This contradicts the assumption that $d_Y(f(p_n), f(q_n)) > \varepsilon$ for some $\varepsilon > 0$. ($\lightning$)

$\square$

11. Suppose f is a uniformly continuous mapping of a metric space X into a metric space Y and prove that $\{f(x_n)\}$ is a Cauchy sequence in Y for every Cauchy sequence in X. Use this result to give an alternative proof of the theorem stated in Exercise 13.
Proof. Let $\varepsilon > 0$ be arbitrary and let $\{x_n\}$ be a Cauchy sequence in X. Since f is uniformly continuous, there exists a $\delta > 0$ such that

$$|f(x_n) - f(x_m)| < \varepsilon$$

for all $|x_n - x_m| < \delta$. Since $\{x_n\}$ is Cauchy, there exists an $N \in \mathbb{N}$ such that $|x_n - x_m| < \delta$ for all $n, m \geq N$. This same N shows that $\{f(x_n)\}$ is Cauchy in Y.

□

12. A uniformly continuous function of a uniformly continuous function is uniformly continuous. State this more precisely and prove it.
We give the statement and then prove it. Let X, Y, Z be metric spaces. Let $f : X \to Y$ and $g : f(X) \to Z$ be two uniformly continuous functions. Let $h : X \to Z$ be given by

$$h(x) = g(f(x))$$

If f is uniformly continuous on X and g is uniformly continuous on $f(X)$, then h is uniformly continuous on X.
Proof. Let $\varepsilon > 0$ be arbitrary. Since g is uniformly continuous, there exists a $\delta_1 > 0$ such that $d(g(y_1), g(y_2)) < \varepsilon$ for all $d(y_1, y_2) < \delta_1$ and $y_1, y_2 \in f(E)$. Since f is uniformly continuous, there exists a $\delta_2 > 0$ such that $d(f(x_1), f(x_2)) < \delta_1$ for all $x_1, x_2 \in X$ with $d(x_1, x_2) < \delta_2$. Choosing this δ_2, we see that if $x_1, x_2 \in X$ and $d(x_1, x_2) < \delta_2$, then

$$d(g(f(x_1), f(x_2)) < \varepsilon$$

Therefore, $h = g \circ f$ is uniformly continuous.

□

13. Let E be a dense subset of a metric space X, and let f be a uniformly continuous *real* function defined on E. Prove that f has a continuous extension from E to X (See Exercise 5 for terminology).
Proof.

14. Let $I = [0, 1]$ be the closed unit interval. Suppose f is a continuous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.
Proof. Consider the function $g(x) = f(x) - x$. This function is the sum of continuous functions, so it is continuous on I as well. If $g(0) = 0$, then we are finished. Suppose that $g(0) \neq 0$. Then we must have $g(0) = f(0) > 0$. On the other hand, $g(1) = f(1) - 1 \leq 0$. If $g(1) = 0$, then $f(1) = 1$ and we are done. Otherwise, we must have $g(1) < 0$. Since g is a continuous function on a closed interval, and since $g(0) > 0 > g(1)$, by Theorem 4.23, there exists some $c \in (0, 1)$ satisfying $g(c) = 0$. Therefore, $f(c) = c$.

■

15. Call a mapping X into Y open if $f(V)$ is an open set in Y whenever V is an open set in X. Prove that every continuous open mapping of $\mathbb{R}^1$ into $\mathbb{R}^1$ is monotonic.
Proof. Suppose that f is continuous and not monotonic. Without loss of generality, it is possible to find a local maximum $y \in X$ and a neighborhood (x, z) such that f is monotonically increasing on (x, y) and monotonically decreasing on (y, z). Let $d = \inf\{f(s) : s \in (x, z)\}$. We have $f((x, z)) = (d, f(y)]$, which is not an open set. Thus, f is not an open map.

■

16. Let $[x]$ denote the largest integer contained in x, that is, $[x]$ is the integer such that $x - 1 < [x] \leq x$, and let $(x) = x - [x]$ denote the fractional part of x. What discontinuities do the functions $[x]$ and (x) have?
Solution.

Chapter 5: Differentiation

1. Let f be defined for all real x, and suppose that

$$|f(x) - f(y)| \le (x-y)^2$$

for all real x and y. Prove that f is constant.

Proof. Fix a $y \in \mathbb{R}$. Let $\varepsilon > 0$ be arbitrary. Choose $\delta = \varepsilon$ and consider $x \in \mathbb{R}$ satisfying $|x-y| < \delta$. Taking absolute values on both sides of the above condition and rearranging, we see that

$$\left| \frac{f(x)-f(y)}{x-y} - 0 \right| \le |x-y| < \delta = \varepsilon$$

Hence

$$f'(y) = \lim_{x \to y} \frac{f(x)-f(y)}{x-y} = 0$$

Since this argument applies to any $y \in \mathbb{R}$, we see that $f'(y) = 0$ for all $y \in \mathbb{R}$. Therefore, by Theorem 5.11, f is constant.

$\square$