Geometric Anatomy of Theoretical Physics

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May 2024

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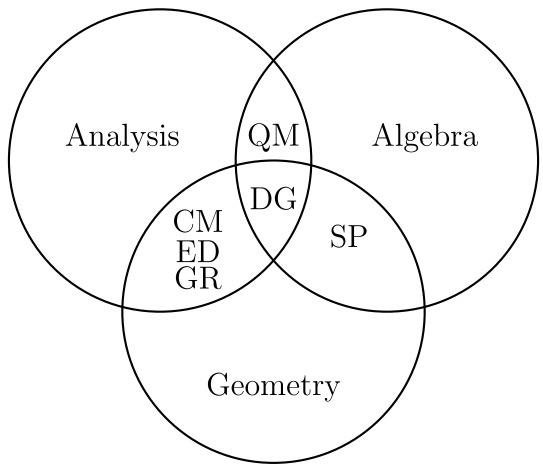
0.1 Preface

The aim of this course is to provide the proper mathematical language for:

1. Classical mechanics (CM)
2. Electromagnetism
3. Quantum mechanics (QM)
4. Statistical physics (SP)

We can broadly view many fields in physics as lying on the intersection of several mathematical disciplines.



where electrodynamics (ED), general relativity (GR), and differential geometry (DG) are abbreviated above. This course focuses on differential geometry.

0.2 Structure of the Course

The course is structured in a way where we discuss concepts in a way such that they build on top of each other.

1. Any space, such as the space of configurations in classical mechanics, the phase space, or the physical space over which we construct our Hilbert spaces in quantum mechanics, involve some set of points. In some sense, this is the coarsest structure of such a space.
2. To describe the axioms of set theory, we need first to describe propositional and predicate logic.

3. Next, another important notion is continuity, such as the continuity of the path over which a particle is travelling, for this we need to be able to distinguish subsets of the set we have. This leads to topology.

4. Due to the abundance of topological spaces, we will need to limit the scope of our discussion to ones of interest to us. Our concern will be topological manifolds, which are spaces that look like $\mathbb{R}^d$ locally at some point.

5. We also need to be able to discuss a notion of change, of differentiability. So this leads to differentiable manifolds and charts.

6. Bundles involve attaching manifolds to other manifolds at some point. Much of modern physics may be cast into the language of bundles.

7. Geometry comes in the form of tensor fields on the structure of bundles. This may be sympletic geometry or metric geometry.

8. Finally, we have CM, EM, QM, SP, SR/GR, the physical concepts, sits on top of these mathematics.

The above structure where the dependencies go from bottom upwards and we conceptualize each notion as a building block for other notions, is shown below.

Physics
Geometry
Bundles
Diff. Manifolds
Top. Manifolds
Topology
Set Theory
Logic

1 Axiomatic Set Theory

1.1 Propositional Logic

Definition. Proposition

A proposition p is a variable that can take the values true or false. It may take no other values.

We can build new propositions from given ones. We do this through the use of logical operators. We describe the logical operators now.

1. **unary** operators (4). They consist of the negation, identity, tautology, and contradiction. These are the only possibilities.

p	$\neg p$	$\mathrm{Id}(p)$	Tp	$\perp p$
T	F	T	T	F
F	T	F	T	F

2. **binary** operators ($16 = 2^4$)

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \Rightarrow q$	$p \iff q$
T	T	T	T	F	T	T
T	F	F	T	T	F	F
F	T	F	T	T	T	F
F	F	F	F	F	T	T

Theorem. $(p \implies q) \iff (\neg q \implies \neg p)$

The proof of the theorem follows from routine verification of the truth tables.
Corollary. We can prove assertions by way of contrapositive.
Remark 1. We agree on decreasing binding strength in the sequence $\neg, \wedge, \vee, \implies, \iff$.
Remark 2. All higher order operators $f(p_1, \ldots, p_N)$ can be constructed from a single binary operator, the NAND (not-and) operator, given below. Thus, in reality, the binding strength

p	q	$p \uparrow q$
T	T	F
T	F	T
F	T	T
F	F	T

is simply a convenience.

1.2 Predicate Logic

Definition. Predicate

Informally, a predicate is a proposition-valued function of some variable or variables. A predicate of two variables is called a *relation*.

Example. We may have a proposition $P(x)$ whose truth value depends on what value x takes. So we might think of a mapping from the domain containing x to the values of true and false. $Q(x, y)$ is also a valid predicate of more than one variable.
Strictly speaking, predicate logic is not concerned with determining the origin of x or y, meaning the sets to which they belong, and it is not concerned with how such variables may be compared. We need to leave this open because will define the notion of sets later using this language. We cannot use sets to define the language of logic without being circular. *At this point, we are only dealing with predicates as elementary objects.*
However, we are able to construct new predicates from given ones.

1. We may define
$$Q(x, y, z) :\iff P(x) \wedge R(y, z)$$
Here, we may think of this as defining the value of some proposition $Q(x, y, z)$.

2. We may convert a predicate P of one variable into a proposition. The proposition is
$$\forall x : P(x)$$
The proposition is defined to be true if $P(x) \iff$ true, that is the proposition defined to be true if it is true independent of x. This is a universal quantification.

3. Existence quantification
$$\exists x : P(x) :\iff \neg(\forall x : \neg P(x))$$
Corollary. $\forall x : \neg P(x) \iff \neg(\exists x : P(x))$

4. Quantification for predicates of more than one variable
$$Q(y) :\iff \forall x : P(x, y)$$
In this case, we refer to x as the **bound variable** and y as the **free variable**. The order of quantification in this case matters. This means that $\forall x : \exists y : P(x, y)$ is generically different proposition than $\exists y : \forall x : P(x, y)$.

5. Let $P(x)$ be a predicate. We define the *unique existential quantifier* $\exists!$ by
$$\exists! x : P(x) :\iff (\exists x : P(x)) \wedge \forall y : \forall z : (P(y) \wedge P(z) \implies y = z)$$

An equivalent, shorter definition is

$$\exists! x : P(x) :\iff (\exists x : \forall y : P(y) \iff x = y)$$

1.3 Axiomatic Systems and Theory of Proofs

Definition. Axiomatic System

An axiomatic system is a finite sequence of propositions (or propositional schemes) $a_1, a_2, \ldots, a_N$, which are called **axioms**.

Disclaimer. This explanation is not building the theory up fully from first-order logic or with full rigor, but it gives an interpretation that is functionally the same. For instance, in the above definition, one might object that numbers are used before sets are defined, but we may think of these objects as pre-mathematical numbers.

Definition. Proof

A proof of a proposition p within an axiomatic system $a_1, \ldots, a_N$ is a finite sequence of propositions $q_1, \ldots, q_M = p$ (i.e. the steps of the proof) such that for any $1 \leq j \leq M$ either

(A) q_j is a proposition from the list of axioms.

(T) q_j is a tautology. A tautology is a proposition that is always true regardless of the elementary propositions that compose it (i.e. $p \vee \neg p$ is a tautology).

(M) Modus Ponens. There exists $1 \leq m, n < j$ such that $(q_m \wedge q_n \implies q_j)$ is true.

Remark. If p can be proven from an axiomatic system $a_1, \ldots, a_N$, we often write

$$a_1, \ldots, a_N \vdash p$$

where the $\vdash$ symbol is read as 'proves'.

Remark. This definition allows us to easily recognize a proof. An altogether different matter is to *find* a proof.

Remark. Obviously, any tautology, should it occur in the axioms, can be removed from the list of axioms without impairing the power of the axiomatic system (i.e. if we remove the tautology we can still prove the same things).

An extreme case of this is: the axiomatic system for propositional logic is the empty sequence. This is because in propositional logic, we can only prove tautologies. But we do not need any axioms here since we may always employ a tautology in our proof via axiom (T).

Definition. Consistent

An axiomatic system is *consistent* if there exists a proposition q which cannot be proven from the axioms. In other words

$$\exists q : \neg(a_1, \ldots, a_N \vdash q)$$

Although this definition appears counterintuitive, there is an idea behind it. Consider an axiomatic system containing contradictory propositions:

$$\ldots, s, \ldots, \neg s, \ldots$$

Then by (M) clearly:

$$s \wedge \neg s \implies q$$

but the above is a tautology, since the antecedent of the conditional statement is always false, it follows that the conditional statement itself is always true. The problem is that any statement may be proven if you have contradictory assumptions in the axiomatic systems. So, it is a sign that our axiomatic system is consistent when not every proposition may be proven. This motivates the definition for consistency.

Theorem. Propositional logic is consistent.

Proof. It suffices to show that there exists a proposition that cannot be proven within the propositional logic. Propositional logic has an empty sequence of axioms. So only (T) and (M) must carry any proof. Thus, we may only prove tautologies. Then since $q \wedge \neg q$ is not a tautology, it cannot be proven.

$\square$

Theorem. (Gödel). Any axiomatic system that is powerful enough to encode the elementary arithmetic of natural numbers is either inconsistent or contains a proposition that can neither be proven nor disproven.
This is a very rough and imprecise statement meant to just give an impression.

Note that to disprove means that we can prove the negation.

1.4 The $\in$-Relation

Set theory is built on the postulate that there is a fundamental relation (i.e. a predicate of two variables) called $\in$. There will be no definition of what $\in$ is, or of what a set is. Instead, we will write down 9 axioms which speak of $\in$ and sets.

These axioms are broken up into four groups: basic existence axioms, construction axioms, further existence or construction axioms, and the axiom of foundation. Using the $\in$-relation, we can immediately define

$$x \notin y := \notin (x, y)$$

Also, we define

$$\begin{aligned} x \notin y &:\iff \neg(x \in y) \\ x \subseteq y &:\iff \forall a : (a \in x \implies a \in y) \\ x = y &:\iff (x \subseteq y) \wedge (y \subseteq x) \end{aligned}$$

1.5 Zermelo-Fraenkel Axioms of Set Theory

We list and explain the axioms below. Notice that the axioms never clarify what exactly a set constitutes. Mnemonic: **EE PURP ICF**

1. *Axiom on $\in$-Relation.* $x \in y$ is a proposition if and only if x and y are both sets.

   $$\forall x : \forall y : (x \in y) \veebar \neg(x \in y)$$

   Why is this axiom important? Assume there is some object u that contains all sets that do not contain themselves as an element. To be precise:

   $$\exists u : \forall z : (z \in u \iff z \notin z)$$

   where z is a set.
   Question: Is u a set?
   If u is a set, then by the axiom we must be able to decide whether $u \in u$ is true or false.
   Assume that $u \in u$ is true. By definition, it follows that $u \notin u$ ($\lightning$).
   Assume $u \in u$ is false. Then this is equivalent to saying $u \notin u$. By the definition of u, we have $u \in u$ ($\lightning$). We conclude that u is not a set.

2. *Axiom on existence of an empty set.* There exists a set that contains no elements.

   $$\exists x : \forall y : y \notin x$$

Theorem. There is only one empty set. Call it $\varnothing$.

Proof. We first detail a textbook-style proof. Assume that x and x' are both empty sets. But then

$$\forall y : (y \in x) \implies (y \in x')$$

and

$$\forall y : (y \in x') \implies (y \in x)$$

This means that $x \subseteq x'$ and $x' \subseteq x$, so $x = x'$.

□

Proof. We now give a rigorous, formal proof. We have our assumptions

$$a_1 \iff \forall y : y \notin x, \qquad a_2 \iff \forall y : y \notin x'$$

So, a_1 means that x is an empty set and a_2 means that x' is an empty set. Now, we must write a string of further propositions to reach the statement $x \iff x'$.

$$q_1 \underset{(T)}{\iff} \forall y : y \notin x \implies \forall y : (y \in x \implies y \in x')$$

$$q_2 \underset{(A)}{\iff} \forall y : y \notin x$$

$$q_3 \underset{M(1,2)}{\iff} \forall y : (y \in x \implies y \in x')$$

$$q_4 \underset{(T)}{\iff} \forall y : y \notin x' \implies \forall y : (y \in x' \implies y \in x)$$

$$q_5 \underset{(A)}{\iff} \forall y : y \notin x'$$

$$q_6 \underset{M(4,5)}{\iff} \forall y : (y \in x' \implies y \in x)$$

$$q_7 \underset{M(3,6)}{\iff} x = x'$$

□

3. *Axiom on pair sets.* Let x and y be sets. Then there exists a set that contains as its elements precisely the sets x and y.

   $$\forall x : \forall y : \exists m : \forall u : (u \in m \iff u = x \lor u = y)$$

   We denote this set m by $\{x, y\}$. The order does not matter since we may show

   $$\{x, y\} = \{y, x\}$$

   Remark. $\{x\} := \{x, x\}$.
   So, the pair set axiom provides us with singletons and two element sets.

4. *Axiom on union sets.* Let x be a set. Then there exists a set u whose elements are precisely the elements of the elements of x.

   $$\forall x : \exists u : \forall y : (y \in u \iff \exists s : (y \in s \land s \in x))$$

   The set u is denoted by $\bigcup x$.
   Example. Let a, b, c be sets. Then, by the axiom on pair sets, $\{a\}$ is a set, $\{b\}$ is a set, and $\{\{a\}, \{b\}\}$ is a set. Applying the union set axiom, we obtain a new set

   $$\bigcup x = \{a, b\}$$

Example 2. From the pair set axiom again, we may write a set

$$x = \{\{a\}, \{b, c\}\}$$

Further, by the union set axiom, we know that $\bigcup x$ is a set. We define

$$\{a, b, c\} := \bigcup x$$

Definition. Let $a_1, a_2, \ldots, a_N$ be sets. We define recursively, for $N \geq 3$,

$$\{a_1, \ldots, a_N\} := \bigcup \{\{a_1, \ldots, a_{N-1}\}, \{a_N\}\}$$

For $N < 3$, the set's existence is given instead by the pair set axiom. The idea of the union axiom is that it is a restriction that prevents taking more sets than can "fit into a set," and addresses the concern of Russell's paradox.

5. *Axiom of replacement.* Let R be a functional relation. Let m be a set. Then the image of m under R is again a set.
   Here, a relation R is called functional if

   $$\forall x : \exists! y : R(x, y)$$

   The image of a set m under a function relation R consists of all those y for which there is an $x \in m$ such that $R(x, y)$.
   We are able to write that the image "consists of" or contains all elements y satisfying this functional relation because the axiom guarantees it is a set.
   This is a stronger version of the *principle of restricted comprehension*. The principle of restricted comprehension (PRC) is the following. Let P be a predicate of one variable and let m be a set. Then those elements $y \in m$ for which $P(y)$ holds constitute a set. This set is usually denoted

   $$\{y \in m \mid P(y)\}$$

   The PRC is not to be confused with the inconsistent principle of universal comprehension, which states $\{y \mid P(y)\}$ is a set. The difference is that in the PRC, we must be able to tell that $y \in m$. Further, the restriction in PRC is the condition that $y \in m$, since the resulting set cannot have more elements than m itself.

Aside. The notation $\forall y \in m : P(y)$ is shorthand for

$$\forall y : (y \in m \implies P(y))$$

The notation $\exists y \in m : P(y)$ is shorthand for

$$\exists y : (y \in m \land P(y))$$

Proposition. The axiom of replacement implies the principle of restricted comprehension.

Proof.

Case 1. We have $\neg(\exists y \in m : P(y))$. This is equivalent to $\forall y : y \in m \implies \neg P(y)$. Then

$$\{y \in m \mid P(y)\} := \varnothing$$

Case 2. The complement of Case 1 holds, that is, $\exists \hat{y} \in m : P(\hat{y})$. Then define

$$R(x, y) := (P(x) \wedge (x = y)) \vee (\neg P(x) \wedge (y = \hat{y}))$$

We claim that R is a functional relation. Now we define

$$\{y \in m \mid P(y)\} := \operatorname{im}_R(m)$$

$\square$

Definition. Let $x \subseteq m$. We define the *complement* of a set by

$$m \setminus u := \{x \in m \mid x \notin u\}$$

We may conclude that the complement is a set by the principle of restricted comprehension, which follows from the axiom of replacement.

Definition. Let x be a set. Then we define the *intersection* of x by

$$\bigcap x := \left\{ a \in \bigcup x \,\middle|\, \forall b \in x : a \in b \right\}$$

If $a, b \in x$ and $\bigcap x = 0$, then a and b are said to be *disjoint*.

6. *Axiom on existence of power sets.* Let m be a set. Then there exists a set, denoted $\mathscr{P}(m)$ whose elements are precisely the subsets of m.

   Example. Let $m = \{a, b\}$. Then

   $$\mathscr{P}(m) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$$

   We must have an axiom to guarantee that this is a set. Historically, people incorrectly argued that the power set is a set using the flawed principle of unrestricted comprehension.

7. *Axiom of infinity.* There exists a set that contains the empty set and, together with every other element y, it also contains the set $\{y\}$ as an element. In symbols

   $$\exists x : (\varnothing \in x) \wedge (\forall y : (y \in x \implies \{y\} \in x))$$

   Remark. One such set is, informally speaking, the set with the elements

   $$\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \ldots$$

We can label each of these elements with the natural numbers.

Corollary. $\mathbb{N}$ is a set.

Without this axiom, we could not say that $\mathbb{N}$ is a set.

Remark. As a set, $\mathbb{R} := \mathscr{P}(\mathbb{N})$.

Aside. The version of the axiom of infinity we stated is the first one put forward by Zermelo. A more modern formulation is the following.

Axiom of infinity. There exists a set that contains the empty set and together with every other element y it also contains the set $y \cup \{y\}$ as an element, where

$$x \cup y := \bigcup \{x, y\}$$

With this formulation, the natural numbers look like

$$\mathbb{N} := \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \ldots\}$$

This formulation has two advantages:

1. The natural number n is represented as an n-element set.
2. It generalizes much more naturally to the system of transfinite ordinal numbers where the successor operation $s(x) = x \cup \{x\}$ applies to transfinite ordinals as well as to natural numbers.

8. *Axiom of Choice.* Let x be a set whose elements are non-empty and mutually disjoint. Then, there exists a set y which contains exactly one element of each element of x.

$$\forall x : P(x) \implies \exists y : \forall a \in x : \exists! b \in a : b \in y$$

where

$$P(x) \iff (\exists a : a \in x) \wedge (\forall a : \forall b : (a \in x \wedge b \in x) \implies \bigcap \{a, b\} = \varnothing)$$

Remark. Given a set x, there is no clear prescription of how one picks an element from each element of x.

Remark. The Axiom of Choice is independent of the other eight axioms. Thus, it is possible to have set theory with or without the axiom of choice. We will accept it.

Remark. There are some results that may only be proven from the axiom of choice.

- Every vector space has a basis.
- There exists a complete system of representatives of an equivalence relation.

9. *Axiom of foundation.* Every nonempty set x contains an element y that has none of its elements in common with x. In symbols

$$\forall x : (\exists a : a \in x) \implies \exists y \in x : \bigcap\{x, y\} = \varnothing$$

We may understand this as a non-existence axiom.
Corollary. There is no set that contains itself as an element, that is, $x \in x$ is not true for any set x.

The totality of all these nine axioms are called *ZFC set theory*, which is shorthand for Zermelo-Fraenkel set theory with the Axiom of Choice.

1.6 Classification of Sets

An important exercise to do whenever one defines a notion in mathematics is to ask about the classification. How many different objects of this type do we have? How do we get an overview of this structure? A recurrent theme in mathematics is the classification of *spaces* by means of structure-preserving *maps* between those spaces.
A space usually refers to some set equipped with some structure. A set is a trivial example of a space as it is a space with no other structure beyond that of the set itself.

Definition. Map

A map $\phi : A \to B$ is a relation such that for every $a \in A$, there exists exactly one $b \in B$ such that $\phi(a, b)$.

Notation. We usually denote $\phi : A \to B$ and $a \mapsto \phi(a)$. We can define $\phi(a) := b$ because it is unique. The standard notation obscures the fact that ϕ is a relation.
Terminology. We call

- A the domain of ϕ
- B the target (codomain) of ϕ
- $\phi(A) = \mathrm{im}_\phi(A) := \{\phi(A) \in B \mid a \in A\}$

Definition. Bijection

A map $\phi : A \to B$ is called

- surjective, if $\phi(A) = B$
- injective, if $\phi(a_1) = \phi(a_2) \implies a_1 = a_2$
- bijective, if it is injective and surjective

Definition. Isomorphic Sets

Two sets A and B are called (set-theoretically) isomorphic,

$$A \cong_{\text{set}} B$$

if there exists a bijection $\phi : A \to B$.

Remark. If there is any bijection $A \to B$, then generically, there are many (i.e. bijections are usually not unique).

Classification of sets. A set A is

- infinite if there exists a proper subset $B \subsetneq A$ that is isomorphic to A, $B \cong_{\text{set}} A$. An example would be the $\mathbb{Z}$, since $\mathbb{N} \subsetneq \mathbb{Z}$ and $\mathbb{N} \cong_{\text{set}} \mathbb{Z}$.
  - A is called countably infinite if $A \cong_{\text{set}} \mathbb{N}$.
  - A is called uncountably infinite, otherwise.
- finite, otherwise. In this case, there exists some $N \in \mathbb{N}$ such that $A \cong_{\text{set}} \{1, 2, \ldots, N\}$ and we may write $|A| = N$.

Some more information about maps:

- Given two maps $A \xrightarrow{\phi} B$ and $B \xrightarrow{\psi} C$, one can construct a map $\psi \circ \phi : A \to C$ which assigns
$$a \mapsto \psi(\phi(a))$$
This is called the composition of maps. Diagrammatically, we have
$$\begin{tikzcd} & B \arrow[dr, "\psi"] & \\ A \arrow[ur, "\phi"] \arrow[rr, "\psi \circ \phi"'] & & C \end{tikzcd}$$
We say that the diagram commutes, which means that we arrive at the same location regardless of the path we take.
Obviously, the composition operation $\circ$ is associative:
$$\xi \circ (\psi \circ \phi) = (\xi \circ \psi) \circ \phi$$
we assume that the domain and targets of these maps are compatible.

Definition. Inverse

Let $\phi : A \to B$ be a bijection. Then the inverse of ϕ is the map $\phi^{-1} : B \to A$ defined uniquely by
$$\phi^{-1} \circ \phi = \mathrm{Id}_A \qquad \phi \circ \phi^{-1} = \mathrm{Id}_B$$

The identity map $\mathrm{Id}_M : M \to M$ is a map which takes every element to itself $m \mapsto m$. We require the following diagram to commute:

$$\begin{tikzcd} A \arrow[loop left, "\mathrm{Id}_A"] \arrow[rr, bend left, "\phi"] & & B \arrow[ll, bend left, "\phi^{-1}"] \arrow[loop right, "\mathrm{Id}_B"] \end{tikzcd}$$

Definition. Preimage

Let $\phi : A \to B$ be any map. Then we define the set

$$\mathrm{preim}_\phi(V) := \{a \in A \mid \phi(a) \in V\}$$

where $V \subseteq B$.

If our map is a bijection, then the preimage of an element will always be a singleton set with the element in the domain that maps to that element in the target. Otherwise, the preimage may be more than one element or empty.

1.7 Equivalence Relations

Definition. Equivalence Relation

Let M be a set and $\sim$ a relation such that the following are satisfied:

1. Reflexivity. $\forall m \in M : m \sim m$
2. Symmetry. $\forall m, n \in M : m \sim n \implies n \sim m$
3. Transitivity. $\forall m, n, p \in M : (m \sim n) \wedge (n \sim p) \implies m \sim p$.
   Then $\sim$ iks called an equivalence relation on M.

Examples.

a) $p \sim q :\iff$ 'p is of the same opinion as q'

b) $p \sim q :\iff$ 'p is a sibling of q'
   Since this is not reflexive, it is not an equivalence relation.

c) $p \sim q :\iff$ 'p is taller than q'
   Since this is not symmetric, it is not an equivalence relation.

d) $p \sim q :\iff$ 'p is in love with q'
   This fails all three conditions.

Definition. Equivalence Class

Let $\sim$ be an equivalence relation on M. Then for any $m \in M$ define the set

$$[m] := \{n \in M : m \sim n\}$$

called the equivalence class of m.

Two key properties:

a) $a \in [m] \implies [a] = [m]$. Any element of the equivalence class may act as a representative of the equivalence class.

Proof. Let $x \in [a]$. So $x \sim a$. Since $a \in [m]$, we see that $a \sim m$. By transitivity, $x \sim m$ and $x \in [m]$. Thus, $[a] \subseteq [m]$. Now, let $x \in [m]$. Then $x \sim m$. Since $a \sim m$, by transitivity we have $a \sim x$ so $x \in [a]$ and $[m] \subseteq [a]$. The conclusion follows. □

b) Either $[m] = [n]$ or $[m] \cap [n] = \varnothing$.

Proof. There are two cases. Suppose that $m \sim n$. So we have $n \in [m]$ and from the first property we see that $[m] = [n]$. The remaining case is $m \not\sim n$. Suppose, for contradiction, that $[m] \cap [n] \neq \varnothing$. Then there exists some $x \in [m] \cap [n]$. But then $m \sim x$ and $n \sim x$ and $m \sim n$ ($\lightning$). □

Definition. Quotient Set

Let $\sim$ be an equivalence relation on M. Define the quotient set

$$M/\!\sim \;:= \{[m] \in \mathcal{P}(M) \mid m \in M\}$$

Remark. Due to the axiom of choice, there exists a complete system of representatives for $\sim$, i.e. a set R such that

$$R \cong_{\text{set}} M/\!\sim$$

Remark. Care must be taken when defining maps whose domain is a quotient set and if one uses representatives to define the map.

Example. Consider $M = \mathbb{Z}$ and define

$$m \sim n :\iff m - n \in 2\mathbb{Z}$$

This is an equivalence relation. We know that

$$\ldots = [-4] = [-2] = [0] = [2] = \ldots$$

$$\ldots = [-3] = [-1] = [1] = [3] = \ldots$$

This means that

$$\mathbb{Z}/\sim = \{[0], [1]\}$$

We can think about defining addition on the equivalence classes. On $\mathbb{Z}$ we have regular addition $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$. We wish to inherit an addition:

$$\oplus : {}^{\mathbb{Z}}/_{\sim} \times {}^{\mathbb{Z}}/_{\sim} \to {}^{\mathbb{Z}}/_{\sim}$$

Suppose that we define

$$[a] \oplus [b] := [a + b]$$

Care must be taken precisely because this could be inconsistent (ill-defined). We must check whether the choice of representatives matters. Let a', b' be integers such that $[a'] = [a]$ and $[b'] = [b]$. We have $a - a' \in 2\mathbb{Z}$ and $b - b' \in 2\mathbb{Z}$. As a result, we may write $a - a' = 2m$ and $b - b' = 2n$ for $n, m \in \mathbb{Z}$. So, by definition:

$$[a'] \oplus [b'] = [a' + b'] = [a - 2m + b - 2n] = [a + b - 2(m + n)] = [a + b] = [a] \oplus [b]$$

which shows that the choice of representatives does not matter. This shows that addition is well-defined, in this case.

1.8 Construction of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$

Construction of $\mathbb{N}$

We should be able to construct these important sets from our ZFC axioms. We give an outline. Recall that from the axiom of infinity, we obtain

$$N := \{0, 1, 2, 3, \ldots,\} = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \ldots\}$$

We wish to establish addition on $\mathbb{N}$. We want to do this in a set-theoretic way. We must define some maps on $\mathbb{N}$ to proceed.

1. We define the **successor map**, $S : \mathbb{N} \to \mathbb{N}$, $n \mapsto n \cup \{n\}$.

2. Let $\mathbb{N}^* = \mathbb{N} \setminus \{\varnothing\}$. We also define the **predecessor map** $P : \mathbb{N}^* \to \mathbb{N}$, by choosing the element with the greatest number of elements from a set n.

3. Let $n \in \mathbb{N}$. Define the n^{th} power of S recursively as

$$S^n := S \circ S^{P(n)} \text{ if } n \in \mathbb{N}^* \text{ and}$$
$$S^0 = \mathrm{Id}_{\mathbb{N}}$$

   Essentially this means apply the successor map n times.

4. Finally, we define addition as

$$\begin{aligned} + &: \mathbb{N} \times \mathbb{N} \to \mathbb{N} \\ (m, n) &\mapsto m + n := S^n(m) \end{aligned}$$

With this definition, it is possible to show that addition is commutative and associative. We also have an identity (or neutral element) of 0. Observe that

$$m + 0 = S^0(m) = m \qquad 0 + m = S^m(0) = (S \circ \ldots \circ S)(0) = m$$

We do not have inverses yet, and this motivates the construction of the integers.

Construction of $\mathbb{Z}$

We claim that we may define the integers as

$$\mathbb{Z} := (\mathbb{N} \times \mathbb{N}) /_{\sim}$$

via a suitable equivalence relation $\sim$.
Definition. Let $\sim$ be the relation on $\mathbb{N} \times \mathbb{N}$ given by

$$(m, n) \sim (p, q) :\iff m + q = p + n$$

We check that this is an equivalence relation.
Reflexivity. $m + n = n + m$ by the commutativity of addition. Thus, $(m, n) \sim (m, n)$.
Symmetry. Suppose $(m, n) \sim (p, q)$. Then $m + q = p + n$. But clearly $p + n = m + q$, so $(p, q) \sim (m, n)$. So symmetry follows from the symmetry of equality.
Transitivity. Suppose that $(m, n) \sim (p, q)$ and $(p, q) \sim (r, s)$. Then we have $m + q = p + n$ and $p + s = r + q$. By rearrangement, we have $q = p + s - r$. By substitution, we see that

$$m + q = m + (p + s - r) = p + n$$

But since $m + p + s - r = p + n$, it implies $m + s = r + n$ and so $(m, n) \sim (r, s)$.

Aside. The point of this definition is to somehow capture the idea of negative numbers. The intuition is that (m, n) corresponds to $m - n$. We want to think about when two differences are the same, and this happens precisely when

$$m - n = p - q$$

But by rearrangement, it is just $m + q = p + n$, which is the equivalence relation. It makes sense to define an equivalence class in this way because we can reach a negative integer in many different ways.

Notice that $\mathbb{N} \not\subseteq \mathbb{Z} = {(\mathbb{N} \times \mathbb{N})}/{\sim}$, since $\mathbb{Z}$ contains elements which are equivalence classes of pairs of $\mathbb{N}$. However, it is possible to embed the naturals in the integers, $\mathbb{N} \hookrightarrow \mathbb{Z}$ by the identification

$$n \mapsto [(n, 0)]$$

After mapping elements of n to elements of $\mathbb{Z}$ by this injection, we have a subset of $\mathbb{Z}$.
Definition. Let $n \in \mathbb{N}$. Then

$$-n := [(0, n)] \in \mathbb{Z}$$

Definition. We want to define addition on the integers, $+_{\mathbb{Z}} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, in a way such that it inherits addition from $\mathbb{N}$. We consider

$$[(m, n)] +_{\mathbb{Z}} [(p, q)] := [(m + p, n + q)]$$

We must check that this is well-defined.
Example. We compute

$$\begin{aligned} 2 +_{\mathbb{Z}} (-3) &= [(2, 0)] +_{\mathbb{Z}} [(0, 3)] \\ &= [(2, 3)] \\ &= [(0, 1)] = -1 \end{aligned}$$

Construction of $\mathbb{Q}$

We must define the rationals as a set.

$$\mathbb{Q} := {(\mathbb{Z} \times \mathbb{Z}^*)}/{\sim}$$

where $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$ and $\sim$ is again some suitable equivalence relation.
Definition. Let $\sim$ a relation on $\mathbb{Z} \times \mathbb{Z}^*$ given by

$$(x, y) \sim (u, v) :\Longleftrightarrow \; x \cdot v = u \cdot y$$

The motivation for this definition is similar to that of the integers. The idea of (x, y) is that x is the numerator and y is the denominator. Thus, fractions are pairs of integers under the identification of cancellations.
Again, we may embed $\mathbb{Z} \hookrightarrow \mathbb{Q}$ by

$$x \mapsto [(x, 1)]$$

Definition. Addition on $\mathbb{Q}$, the map $+_{\mathbb{Q}} : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ is defined by

$$[(x, y)] +_{\mathbb{Q}} [(u, v)] := [(x \cdot v + y \cdot u, y \cdot v)]$$

Multiplication on $\mathbb{Q}$, the map $\cdot_{\mathbb{Q}} : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ is defined by

$$[(x, y)] \cdot_{\mathbb{Q}} [(u, v)] = [(x \cdot u, y \cdot v)]$$

where we note that the addition and multiplication on the right hand size are operations over $\mathbb{Z}$.
It is possible to verify that the above addition and multiplication are well-defined.

Construction of $\mathbb{R}$

Finally, we may construct $\mathbb{R}$ from $\mathbb{Q}$. The real numbers are a set defined as

$$\mathbb{R} := {}^{\mathscr{A}}/_{\sim}$$

where $\mathscr{A}$ is the set of almost homomorphisms on $\mathbb{Z}$. There is also the classic construction using Dedekind cuts.

2 Topological Spaces

2.1 Topological Spaces

Establishing a topology on a set gives the weakest structure on that set that allows for notions of continuity (of maps) and convergence (of sequences).

Definition. Topological Space

Let M be some set. Then a choice $\mathcal{O} \subseteq \mathcal{P}(M)$ is called a **topology** on M if

(i) $\varnothing \in \mathcal{O}$ and $M \in \mathcal{O}$.

(ii) If $U, V \in \mathcal{O}$, then
$$\bigcap\{U, V\} \in \mathcal{O}$$

(iii) If we have an arbitrary collection of open sets $C \subseteq \mathcal{O}$, then
$$\bigcup C \in \mathcal{O}$$

The pair $(M, \mathcal{O})$ is called a topological space.
The elements of the topology $\mathcal{O}$ are called open sets.

Remark. Unless $|M| = 1$, there are different topologies $\mathcal{O}$ one can choose on one and the same set.

$\lvert M\rvert$	Number of topologies
1	1
2	4
3	29
4	355
5	6,942
6	209,527
7	9,535,241

Table 1: Number of topologies for different $|M|$

Examples. Let M be any set.

(a) $\mathcal{O} = \{\varnothing, M\}$ is a topology on M. One may verify that it satisfies the definition. This is called the **trivial (or chaotic) topology**.

(b) $\mathcal{O} = \mathcal{P}(M)$ is a topology. This is called the **discrete topology**.

(c) Let $M = \{1, 2, 3\}$. Choose

$$\mathcal{O} = \{\varnothing, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}\}$$

This is closed under arbitrary unions and finite intersections. It is a topology, even though it is not very interesting.

(d) **Key Example.** Let $M = \mathbb{R}^d := \underbrace{\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}}_{d \text{ times}}$. We construct the standard topology, $\mathcal{O}_{\text{std}}$, in three steps.

1. $\forall x \in \mathbb{R}^d$, $\forall r \in \mathbb{R}^+$, we define

$$B_r(x) := \left\{ y \in \mathbb{R}^d : \sqrt{\sum_{i=1}^{d} (y^i - x^i)^2} < r \right\}$$

where $y = (y^1, \ldots, y^d)$ and where we usually write $\|y - x\| = \sqrt{\sum_{i=1}^{d} (y^i - x^i)^2}$. A famous theorem is that regardless of the norm above, the resulting topology will be the same. We call this set $B_r(x)$ the open ball of radius r around the point x.

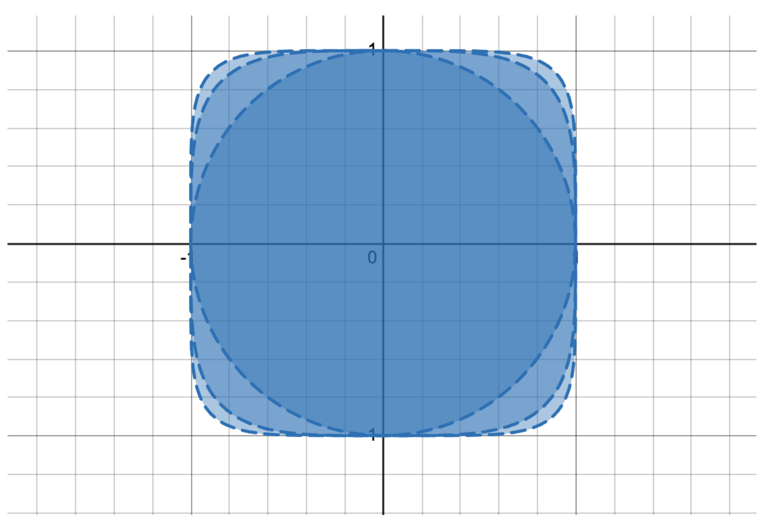
2. We define membership in the standard topology.

$$U \in \mathcal{O}_{\text{std}} :\Longleftrightarrow \forall p \in U : \exists r \in \mathbb{R}^+ : B_r(p) \subseteq U$$

The idea is that U is open if it contains all of its interior points. An interesting point is that if we consider the metric

$$\|y - x\|_{2n} = \left(\sum_{i=1}^{d} (y^i - x^i)^{2n} \right)^{\frac{1}{2n}}$$

we get a shape that approximates a square as $n \to \infty$. We reach a square when using the 'infinity norm' $\| \cdot \|_\infty$.



3. We show that $\mathcal{O}_{\text{std}}$ is a topology.
   (i) We have $\varnothing \in \mathcal{O}_{\text{std}}$. The statement $\forall p : (p \in \varnothing \implies \ldots)$ is true since the premise $p \in \varnothing$ is false. Further, $M = \mathbb{R}^d \in \mathcal{O}_{\text{std}}$. For any point p, we may choose an arbitrary open ball to satisfy the condition.
   (ii) Suppose $U, V \in \mathcal{O}_{\text{std}}$. Consider an arbitrary point $p \in (U \cap V)$. Since $p \in U$, there exists some $r_1 > 0$ such that $B_{r_1}(p) \subseteq U$. Similarly, since $p \in V$, there exists some $r_2 > 0$ such that $B_{r_2}(p) \subseteq V$. Without loss of generality, suppose that $r_2 > r_1$. Then $B_{r_1}(p) \subseteq B_{r_2}(p) \subseteq V$ and so $B_{r_1}(p) \subseteq (U \cap V)$. We can also take $\min(r_1, r_2)$ as the radius to show this. We conclude that $U \cap V \in \mathcal{O}_{\text{std}}$.
   (iii) Let $C \subseteq \mathcal{O}_{\text{std}}$. Let $p \in \bigcup C$. Then, there exists some $U \in C$ such that $p \in U$. Since $U \in \mathcal{O}_{\text{std}}$, there exists an $r > 0$ such that $B_r(p) \subseteq U \subseteq \bigcup C$. So, using this same r, $\bigcup C \in \mathcal{O}_{\text{std}}$.

□

2.2 Construction of New Topologies from Given Topologies

Definition. Induced Topology on a Subspace

Let $(M, \mathcal{O})$ be a topological space. Let $N \subsetneq M$. Then

$$\mathcal{O}\big|_N := \{U \cap N : U \in \mathcal{O}\} \subseteq \mathcal{P}(N)$$

is a topology on N called the induced (subspace) topology.

Proposition. The induced topology is in fact a topology.

Proof. We verify the conditions.

(i) Since $\varnothing \cap N = \varnothing$, it follows that $\varnothing \in \mathcal{O}\big|_N$. Since $N \cap M = N$, it follows that $N \in \mathcal{O}\big|_N$.

(ii) Suppose $S, T \in \mathcal{O}\big|_N$. By definition, there exists $U, V \in \mathcal{O}$ such that $S = U \cap N$ and $T = V \cap N$. Observe that

$$S \cap T = (U \cap N) \cap (V \cap N) = (U \cap V) \cap N$$

which shows that $S \cap T \in \mathcal{O}\big|_N$

(iii) Suppose $C := \{S_\alpha \mid \alpha \in \mathcal{A}\} \subseteq \mathcal{O}\big|_N$. Observe

$$\bigcup_{\alpha \in \mathcal{A}} S_\alpha = \bigcup_{\alpha \in \mathcal{A}} U_\alpha \cap N = \left(\bigcup_{\alpha \in \mathcal{A}} U \right) \cap N$$

Thus, $\bigcup C \in \mathcal{O}\big|_N$. Note that for all $\alpha \in \mathcal{A}$ there exists $U_\alpha \in \mathcal{O}$ such that $S_\alpha = U_\alpha \cap N$ since $S_\alpha \in \mathcal{O}\big|_N$

□

Example. Consider $(\mathbb{R}, \mathcal{O}_{\text{std}})$. Let

$$N = [-1, 1] := \{x \in \mathbb{R} : -1 \leq x \leq 1\}$$

Consider the induced topology $(N, \mathcal{O}_{\text{std}}\big|_N)$. We claim that $(0, 1] \notin \mathcal{O}_{\text{std}}$ but

$$(0, 1] = (0, 2) \cap [-1, 1] \in \mathcal{O}_{\text{std}}\big|_N$$

The point is that whether or not a set is open depends on the topology in which that set is being considered.

Definition. Quotient Topology

Let $(M, \mathcal{O}_M)$ be a topological space and let $\sim$ be an equivalence relation on M. Then, the quotient set:

$$M/\sim \ = \{[m] \in \mathcal{P}(M) \mid m \in M\}$$

can be equipped with the *quotient topology* $\mathcal{O}_{M/\sim}$ defined by:

$$\mathcal{O}_{M/\sim} := \left\{ U \mid (U \subseteq M/\sim) \wedge \left(\bigcup U = \bigcup_{[a] \in U} [a] \in \mathcal{O} \right) \right\}$$

An equivalent definition of the quotient topology is as follows. Let $q : M \to M/\sim$ be the map

$$\begin{aligned} q : M &\to M/\sim \\ m &\mapsto [m] \end{aligned}$$

Then we have

$$\mathcal{O}_{M/\sim} := \{U \mid (U \subseteq M/\sim) \wedge (\text{preim}_q(U) \in \mathcal{O})\}$$

Definition. Closed

Let $(M, \mathcal{O})$ be a topological space. The set $C \subseteq M$ is called closed if its complement $M \setminus C$ is open.

Example. $[0, 1]$ is closed in $(\mathbb{R}, \mathcal{O}_{\text{std}})$ because $R \setminus [0, 1] = (-\infty, 0) \cup (1, \infty) \in \mathcal{O}_{\text{std}}$.

⚠ In general, a subset of a topological space can be:

(i) open

(ii) closed

(iii) open and closed

(iv) open and not closed

(v) not open and closed

(vi) not open and not closed

So, the openness and closedness of a set have nothing to do with each other, except via the complement. Notice that both the empty set and the entire set M are open and closed.

After defining the notion of a connected topological space, we will be able to show that in a connected topological space, the only sets that are both closed and open are the empty set and the entire set.

Example. Consider the set

$$S^1 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

One way to establish a topology on S^1 is to let

$$\mathcal{O} := \mathcal{O}_{\text{std}}\big|_{S^1}$$

From this, we can see that an arc on the circle with open endpoints is an open set (it is the intersection of some circle with S^1) and that the set consisting of a single point on the circle is not open.

Another way to define a topology on the circle is to consider

$$S^1 = \mathbb{R} /\sim$$

with the equivalence relation

$$x \sim y :\Longleftrightarrow \exists n \in \mathbb{Z} : y = x + 2\pi n$$

Then the circle can be defined as the set S^1 equipped with the quotient topology.

Definition. Product Topology

Let $(A, \mathcal{O}_A)$ and $(B, \mathcal{O}_B)$ be two topological spaces. We may equip $A \times B$ with the product topology, $\mathcal{O}_{A\times B}$ defined implicitly by:

$$U \in \mathcal{O}_{A\times B} :\Longleftrightarrow \forall p \in U : \exists S \in \mathcal{O}_A, T \in \mathcal{O}_B : S \times T \subseteq U$$

where $p = (a, b) \in A \times B$, $a \in S$, and $b \in T$.

Remark.

1. We can do this for any finite Cartesian product $A_1 \times A_2 \times \ldots \times A_n$.
2. $\mathcal{O}_{\text{std } \mathbb{R}^d} = \mathcal{O}_{\mathbb{R}\times\mathbb{R}\times\ldots\times\mathbb{R}}$
   The reason for this is because the topologies generated by circles and boxes are the same (i.e. we can place a box in a circle and vice versa).

Topology is not only useful for physical spaces and applications, but may have many diverse applications depending on the topology with which a space is equipped. For instance, Furstenberg discovered a topological proof of the infinitude of primes based on choosing a topology on $\mathbb{Z}$.

2.3 Convergence

Definition. Converge

A sequence q (i.e. a map $q : \mathbb{N} \to M$) on a topological space $(M, \mathcal{O})$ is said to converge against a limit point $a \in M$ if

$$\forall U_a \in \mathcal{O} : \exists N \in \mathbb{N} : \forall n > N : q(n) \in U_a$$

We often refer to U_a as an open neighborhood of a.

Example.

(a) Consider the topological space $(M, \{\varnothing, M\})$. Let $q : \mathbb{N} \to M$ be some sequence. We claim that any sequence converges against every point. This is why it is called the chaotic topology and why it is not useful.

(b) Consider M equipped with the discrete topology, $(M, \mathcal{P}(M))$. In this topology, only the almost constant sequences converge. Almost constant means that the sequence is constant for all but a finite number of values. This result is immediate since the singleton set $\{a\} \in \mathcal{P}(M)$ is considered open.

(c) Consider the topological space $(\mathbb{R}, \mathcal{O}_{\text{std}})$.
Theorem. Let $q : \mathbb{N} \to \mathbb{R}$ be a sequence. The sequence q converges against $a \in \mathbb{R}^d$ if

$$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall n > N : \|q(n) - a\| < \varepsilon$$

2.4 Continuity

Definition. Continuity

Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ be topological spaces. Let $\phi : M \to N$ be a map. Then ϕ is called continuous if

$$\forall V \in \mathcal{O}_N : \text{preim}_\phi(V) \in \mathcal{O}_M$$

For brevity, we may say that ϕ is continuous if preimages of open sets are open.

In the above definition, note how openness differs depending on the topology.
Examples.

(a) Consider $\phi : M \to N$. Choose $\mathcal{O}_M = \mathcal{P}(M)$, the discrete topology for the domain. We claim that any map ϕ is continuous regardless of the choice of the topology of the codomain. The preimage of any open set in $\mathcal{O}_N$ will be a subset of M, by definition. Since it is a subset, it is an element of the discrete topology and is open. Thus, ϕ is continuous. This is why we never equip the domain with the discrete topology if we want continuity to have any meaning.

(b) Let $\phi : M \to N$ be a map and consider $\mathcal{O}_N = \{\varnothing, N\}$. Then, any map is continuous. The preimage of the empty set is the empty set, which is open. Observe that

$$\mathrm{preim}_\phi(N) = \{m \in M : \phi(m) \in N\} = M \in \mathcal{O}_M$$

showing that the preimage of N is open in any topology $\mathcal{O}_M$.

(c) Let $\phi : \mathbb{R}^d \to \mathbb{R}^f$ be a map, where $\mathbb{R}^d$ and $\mathbb{R}^f$ are each equipped with the standard topology. From this, we may recover the standard definition of continuity between these spaces.

Definition. Homeomorphism

Let $\phi : M \to N$ be a bijection. Now equip these sets with a topology, $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$. We call ϕ a homeomorphism if

1. $\phi : M \to N$ is continuous
2. $\phi^{-1} : N \to M$ is continuous

Remark: Homeomorphisms are the structure-preserving maps in topology. If there exists a homeomorphism ϕ between $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$,

$$\begin{tikzcd} M \arrow[r, bend left, "\phi"] & N \arrow[l, bend left, "\phi^{-1}"] \end{tikzcd}$$

then ϕ provides a one-to-one pairing of the open sets of M with the open sets of N. If there exists a homeomorphism between two topological spaces $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$, we say that the two spaces are *homeomorphic* or *topologically isomorphic* and we write

$$(M, \mathcal{O}_M) \cong_{\text{top}} (N, \mathcal{O}_N)$$

Of course, this implies that $M \cong_{\text{set}} N$.

2.5 Topological Properties I: Separation Properties

There is no complete list of topological properties that would allow us to conclude that spaces satisfying those properties would be homeomorphic.

Definition. T1

A topological space $(M, \mathcal{O})$ is called T1 if for any two distinct points $p, q \in M$, $p \neq q$:

$$\exists U_p \in \mathcal{O} : q \notin U_p$$

where U_p is a $U \in \mathcal{O}$ such that $p \in U$.

Definition. Hausdorff (T2)

A topological $(M, \mathcal{O})$ is said to be *T2* or *Hausdorff* if for any two distinct points, there exist non-intersecting open neighbourhoods of these two points:

$$\forall p, q \in M : p \neq q \implies \exists U_p, V_q \in \mathcal{O} : U_p \cap V_q = \varnothing$$

Notice how being T2 is a stronger condition. There are examples of sets that are T1 but not T2.

Remark. There are many other separation axioms. For instance, T2.5 is when we can find a closed ball around one of the points. There are also T3 and T4 classifications. These are stronger and stronger conditions.

Examples.

- The topological space $(\mathbb{R}^d, \mathcal{O}_{\text{std}})$ is T2 and hence also T1.
- The Zariski topology on an algebraic variety is T1 but not T2.
- The topological space $(M, \{\varnothing, M\})$ does not have the T1 property since for any $p \in M$, the only open neighbourhood of p is M and for any other $q \neq p$ we have $q \in M$. Moreover, since this space is not T1, it cannot be T2 either.

2.6 Compactness and Paracompactness

Compact spaces are useful to get a grasp on the objects with which we are working. It is often the case that theorems will be proven in compact spaces before seeing if they apply more generally in the non-compact case.

Definition. Compact

A topological space $(M, \mathcal{O})$ is called *compact* if every open cover C has a finite subcover $\widetilde{C}$.

Observe that compactness is a topological property. What we mean by topological property is any property which is defined only in terms of the topology. In particular, two homeomorphic topological spaces, they share these topological properties. Often, we can check compactness by showing compactness in one space and then constructing a homeomorphism between that space and the space whose compactness we want to check.

Aside. Let $C \subseteq \mathcal{O}$. Then C is called an open cover if

$$\bigcup C = M$$

A finite subcover $\widetilde{C}$ is a cover which is a finite subset of the cover:

$$\widetilde{C} \subseteq C \text{ and } |\widetilde{C}| < \infty$$

Definition. Compactness of Subspace

Let $(M, \mathcal{O})$ be a topological space. A subset $N \subseteq M$ is called compact if $(N, \mathcal{O}\big|_N)$ is compact.

Example. Consider the subspace topology $[0,1]$ relative to $(\mathbb{R}, \mathcal{O}_{\text{std}})$. This is compact. However, the subspace topology $(0,1)$ is not.

Theorem. (Heine-Borel). In a *metric space* (M, d), equipped with the *metric-induced topology*, every closed and bounded subset of M is compact.

Definition. *Bounded.* A subset S of $\mathbb{R}^d$ is said to be bounded if

$$\exists r \in \mathbb{R}^+ : S \subseteq B_r(0)$$

The ball does not need to be centered at 0, the point is that there is some open set with finite diameter which contains the subset.

Definition. *Metric.* Let M be a set. A metric is a map $d : M \times M \to \mathbb{R}_{\geq 0}$ satisfying

(a) $d(m,n) = d(n,m)$

(b) $d(m,n) \geq 0$ and $d(m,n) = 0 \iff m = n$

(c) $d(a,b) + d(b,c) \geq d(a,c)$

Definition. *Metric-induced Topology.* The idea is similar to that of the standard topology on $\mathbb{R}$, except the balls are now defined more generally. Let $r \in \mathbb{R}^+$. Define

$$B_r(p) := \{q \mid d(p,q) < r\}$$

These balls generate the topology $\mathcal{O}_d$ on M by:

$$U \in \mathcal{O}_d :\iff \forall p \in U : \exists r \in \mathbb{R}^+ : B_r(p) \subseteq U$$

In other words, we use these balls to define the interior point of a set.

Aside. French Railroad Metric. This metric is given by:

$$d(a,b) = \begin{cases} |a-b| & \text{if } a = cb \text{ for some } c \in \mathbb{R} \\ |a| + |b| & \text{otherwise} \end{cases}$$

The point is this is an example of a metric which is not induced by a norm, so it does not satisfy our intuitions associated with the Euclidean norm.

Examples of Compact Sets.

(a) $[0,1]$ is compact. It is closed and bounded so it is compact by Heine-Borel.

(b) $\mathbb{R}$ is not compact.

Proof. Consider the open cover

$$\{(n, n+2) : n \in \mathbb{Z}\}$$

Consider any finite subset S of this open cover. Since the subset is finite, we may choose

$$a = \max\{n \in \mathbb{Z} : (n, n+2) \in S\}$$

Then, it follows that $a+2$ is not covered. This shows that no finite subset covers $\mathbb{R}$. □

Theorem. Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ be topological spaces that are compact. Then, the product topology $(M \times N, \mathcal{O}_{M \times N})$ is again compact. This means the finite Cartesian product of compact spaces is again compact with the product topology.

However, compactness is a very strong requirement, and sometimes it does not hold. We have a weaker notion of paracompactness.

Definition. Paracompact

A topological space $(M, \mathcal{O})$ is paracompact if every open cover C has a open refinement $\widetilde{C}$ that is locally finite.

Definition. *Refinement.* Let $(M, \mathcal{O})$ be a topological space. Let C be a cover. An refinement of C is a cover R such that

$$\forall U \in R : \exists V \in C : U \subseteq V$$

Any subcover of a cover is a refinement of that cover, but the converse is not true in general. A refinement R is said to be:

- *open* if $R \subseteq \mathcal{O}$;

- *locally finite* if for any $p \in M$ there exists a neighbourhood $U(p)$ such that the set:

$$\{U \in R \mid U \cap U(p) \neq \varnothing\}$$

is finite.

Corollary. If a space is compact, then it is paracompact.

Proof. Suppose that the space is compact. Then for every open cover C there is a finite subcover $\widetilde{C}$. The finite subcover is a refinement because it is a subset of the cover. Finally, because the subcover is finite, any point can only be covered a finite number of times, which shows that it is locally finite. Hence, it is paracompact using the same subcover.

□

Theorem. Stone's Metrization Theorem.
Every metrizable topological space is paracompact.

Metrizable means that the space may be equipped with a metric (which then induces a topology that is the same one of the space).

Examples.

(a) Consider the topological space $(\mathbb{R}^d, \mathcal{O}_{\text{std}})$. We may equip it with the Euclidean norm. Then, this space is paracompact.

(b) Alexandrov line or long line. It is not paracompact. The premise is that we may think of the real line as a countable sequence of intervals $[n, n+1)$. So we say $\mathbb{R} = \mathbb{Z} \times [0, 1)$. Now, consider

$$L = \omega_1 \times [0, 1)$$

where ω_1 is a non-countable set. This notation is from the theory of ordinal numbers, ω_1 is the first non-finite ordinal number. This construction L is not paracompact.

Theorem. Let $(M, \mathcal{O}_M)$ be a paracompact space and let $(N, \mathcal{O}_N)$ be a compact space. Then $M \times N$ (equipped with the product topology) is paracompact.

Corollary. Let $(M, \mathcal{O}_M)$ be a paracompact space and let $(N_i, \mathcal{O}_{N_i})$ be compact spaces for every $1 \leq i \leq n$. Then $M \times N_1 \times \cdots \times N_n$ is paracompact.

Theorem. Let $(M, \mathcal{O}_M)$ be a Hausdorff topological space. Then, it is paracompact if and only if every open cover C admits a partition of unity subordinate to that cover.

Definition. Partition of Unity

Let $(M, \mathcal{O}_M)$ be a topological space. A *partition of unity* is a set $\mathcal{F}$ of continuous functions $f : M \to [0,1]$ such that for each $p \in M$ the following conditions hold:

1. there exists U_p such that the set
$$\{f \in \mathcal{F} \mid \forall x \in U_p : f(x) \neq 0\}$$
is finite;
2. the sum over functions $f \in \mathcal{F}$ at p is 1
$$\sum_{f \in \mathcal{F}} f(p) = 1$$

Aside. The support of a real-valued function is the closure of the subset of the domain consisting of elements which are not mapped to zero.

If C is an open cover, then $\mathcal{F}$ is said to be subordinate to the cover C if:

$$\forall f \in \mathcal{F} : \exists U \in C : f(x) \neq 0 \implies x \in U$$

The motivation is for integration on manifolds. The outlook is we have a cover and charts, and on one such open set of the cover, we can go to local coordinates. However, there may be overlap in the open sets that are mapped, and we need this partition of unity to deal with this overlapping areas when integrating.

Example. We have shown previously that $(\mathbb{R}, \mathcal{O}_{\text{std}})$ is paracompact using Stone's theorem. From the theorem above, then, every open cover should admit a partition of unity. As a simple example, consider $\mathcal{F} = \{f, g\}$, where:

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x^2 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x \geq 1 \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ 1 - x^2 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } x \geq 1 \end{cases}$$

Then $\mathcal{F}$ is a partition of unity of $\mathbb{R}$. Indeed, $f, g : \mathbb{R} \to [0,1]$ are both continuous, condition i) is satisfied since $\mathcal{F}$ itself is finite, and we have $\forall\, x \in \mathbb{R} : f(x) + g(x) = 1$.
Let $C := \{(-\infty, 1), (0, \infty)\}$. Then C is an open cover of $\mathbb{R}$ and since:

$$f(x) \neq 0 \implies x \in (0, \infty) \qquad \text{and} \qquad g(x) \neq 0 \implies x \in (-\infty, 1),$$

the partition of unity $\mathcal{F}$ is subordinate to the open cover C.

2.7 Connectedness and Path-Connectedness

These are useful concepts because they tell us whether a topological space consists of one piece or several pieces.

Definition. Connected

A topological space $(M, \mathcal{O})$ is called connected unless there exist nonempty, non-intersecting open sets A and B such that $M = A \cup B$

Example. Consider $(\mathbb{R} \setminus \{0\}, \mathcal{O}_{\mathbb{R}\setminus\{0\}})$ where we have equipped the punctured real line with its subspace topology. This set is not connected: let $A = (-\infty, 0)$ and $B = (0, \infty)$ both are nonempty, non-intersecting, and open in the induced topology. Finally, $\mathbb{R} \setminus \{0\} = A \cap B$.

Theorem. The interval $[0, 1] \subseteq \mathbb{R}$ equipped with the induced topology is connected.

Theorem. A topological space $(M, \mathcal{O})$ is connected if and only if the $\varnothing$ and M are the only subsets which are both open and closed.

Proof. Suppose, for contradiction, $(M, \mathcal{O})$ is connected and there exists a set $U \subseteq M$ that is open and closed and that is neither the empty set nor M. Then we may write

$$M = U \cup (M \setminus U)$$

Since U is closed, by definition, $(M \setminus U)$ is open. Further, $M \setminus U$ is nonempty, if not, then $U = M$. By construction, this union is disjoint. Hence, $(M, \mathcal{O})$ is not connected ($\lightning$).
Now, suppose that $(M, \mathcal{O})$ is not connected. Then there exists nonempty, disjoint open sets A, B such that

$$M = A \cup B = A \cup (M \setminus A)$$

Since A is open, $M \setminus A$ is closed. Hence, $M \setminus A$ is a set that is not M or $\varnothing$ which is both open and closed.

$\square$

Definition. Path-connected

A topological space $(M, \mathcal{O})$ is called path-connected if for every pair of points $p, q \in M$ there exists a continuous curve $\gamma : [0, 1] \to M$ such that $\gamma(0) = p$ and $\gamma(1) = p$.

Examples.

(a) $(\mathbb{R}^d, \mathcal{O}_{\text{std}})$ is path-connected. We can take the straight line homotopy between any two points.

(b) Define

$$S := \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \mid x \in (0,1] \right\} \cup \{(0,0)\} \subseteq \mathbb{R}^2$$

The induced space $(S, \mathcal{O}_{\text{std}}\big|_S)$ is connected but not path-connected.

Theorem. If a space is path-connected, then it is connected.

Proof. Let $(M, \mathcal{O})$ be a topological space. Suppose, for contradiction, that M is path-connected but not connected. Since M is not connected, there exist nonempty, disjoint, open sets A, B such that

$$M = A \cup B$$

Since A and B are nonempty, we may choose $a \in A$ and $b \in B$. Since $(M, \mathcal{O})$ is path-connected, this means there exists a continuous curve $\gamma : [0,1] \to M$ such that $\gamma(0) = a$ and $\gamma(1) = b$. However, notice that

$$[0,1] = \text{preim}_\gamma(M) = \text{preim}_\gamma(A \cup B) = \text{preim}_\gamma(A) \cup \text{preim}_\gamma(B)$$

Since γ is continuous, we see that $\text{preim}_\gamma(A)$ and $\text{preim}_\gamma(B)$ are both open. Also, $0 \in \text{preim}_\gamma(A)$ and $1 \in \text{preim}_\gamma(B)$, so both sets are nonempty. Since $A \cap B = \varnothing$, the preimages are also disjoint. Hence, $[0,1]$ is not connected ($\lightning$).

$\square$

2.8 Homotopic Curves and the Fundamental Group

So far, we have considered topological properties of a yes or no character: is it Hausdorff, is it compact, is it connected? Now, we define 'group-valued' properties. If two topological spaces are associated with different groups, they cannot be homeomorphic.

Definition. Homotopic

Let $(M, \mathcal{O})$ be a topological space. Two curves $\gamma : [0,1] \to M$ and $\delta : [0,1] \to M$ with $\delta(0) = \gamma(0)$ and $\delta(1) = \gamma(1)$ are called *homotopic* if there exists a continuous map

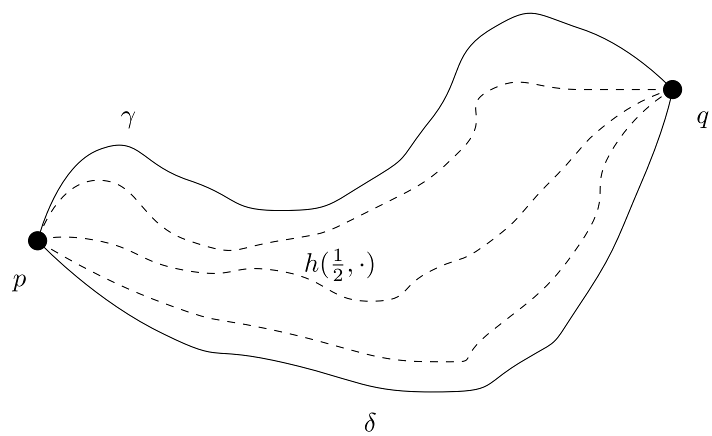
$$h : [0,1] \times [0,1] \to M$$

such that $h(0, \lambda) := \gamma(\lambda)$ and $h(1, \lambda) := \delta(\lambda)$ for $\lambda \in [0,1]$.

We can think of the first parameter of h as interpolating between the curves, and the second parameter as travelling along the curve.

Remark. Observe that $[0,1] \times [0,1]$ is a topological space from several steps. First, $[0,1]$ is a topological space with the induced topology from the standard topology on $\mathbb{R}$. Next, the

Cartesian product $[0,1] \times [0,1]$ has the product topology.
Pictorially, two curves are homotopic if they can be *continuously deformed* into one another.



Definition. Equivalence of Curves

We call two curves γ, δ equivalent if they are homotopic.

$$\gamma \sim \delta :\Longleftrightarrow \gamma \text{ is homotopic to } \delta$$

Fact. Homotopy of curves is an equivalence relation.

Definition. The Space of Loops, $\mathscr{L}_p$

Let $(M, \mathcal{O})$ be a topological space. Then for every $p \in M$, we define the space of loops at p by

$$\mathscr{L}_p := \{\gamma : [0,1] \to M : \gamma \text{ is continuous }, \gamma(0) = \gamma(1) = p\}$$

Definition. Concatenation

We define the concatenation operation $*_p : \mathscr{L}_p \times \mathscr{L}_p \to \mathscr{L}_p$ by

$$(\gamma * \delta)(\lambda) := \begin{cases} \gamma(2\lambda) & \text{if } 0 \leq \lambda \leq \frac{1}{2} \\ \delta(2\lambda - 1) & \text{if } \frac{1}{2} \leq \lambda \leq 1 \end{cases}$$

The intuition of the concatenation operator is that we travel both paths of the two loops for $\lambda \in [0,1]$.

Definition. Fundamental Group

Let $(M, \mathcal{O})$ be a topological space. The *fundamental group* $(\pi_1(p), \bullet)$ of $(M, \mathcal{O})$ at $p \in M$ is the set

$$\pi_1(p) := \mathcal{L}_p /_\sim = \{[\gamma] : \gamma \in \mathcal{L}_p\}$$

where $\sim$ is the homotopy equivalence relation, together with the map

$$\begin{aligned} \bullet : \pi_1(p) \times \pi_1(p) &\to \pi_1(p) \\ (\gamma, \delta) &\mapsto [\gamma] \bullet [\delta] := [\gamma * \delta] \end{aligned}$$

Remark. The intuitive idea of the homotopy relation $\sim$ is clear. In effect, we ignore continuous deformations of loops.

Aside. Recall that a group is a pair $(G, \bullet)$ where G is a set and $\bullet : G \times G \to G$ is a map (also called a *binary operation*) such that

(i) $\forall a, b, c \in G : (a \bullet b) \bullet c = a \bullet (b \bullet c)$

(ii) $\exists e \in G : \forall g \in G : g \bullet e = e \bullet g = g$

(iii) $\forall g \in G : \exists g^{-1} \in G : g \bullet g^{-1} = g^{-1} g = e$

A group is called *abelian* (or *commutative*) if for all $a, b \in G$, $a \bullet b = b \bullet a$, in addition to the above conditions.
A *group isomorphism* between two groups $(G, \bullet)$ and $(H, \circ)$ is a bijection $\phi : G \to H$ such that

$$\forall a, b \in G : \phi(a \bullet b) = \phi(a) \circ \phi(b)$$

If there exists a group isomorphism between $(G, \bullet)$ and $(H, \circ)$, then we say that G and H are isomorphic in the group theoretic sense, and we write $G \cong_{\mathrm{grp}} H$

We now verify that the $\bullet$ operation associated with the fundamental group, that is concatenation, satisfies the group axioms. We remark that because we are considering equivalence classes of curves based on homotopy, the speed at which the curve travels does not matter. First, the operation $\bullet$ is associative, this verification is straightforward. The neutral element of the fundamental group $(\pi_1(p), \bullet)$ is the equivalence class of the constant curve γ_e, defined by:

$$\begin{aligned} \gamma_e : [0, 1] &\to M \\ \lambda &\mapsto \gamma_e(0) = p \end{aligned}$$

Finally, for each $[\gamma] \in \pi_1(p)$, the inverse under $\bullet$ is the element $[-\gamma]$ where $-\gamma$ is defined by:

$$\begin{aligned} -\gamma : [0,1] &\to M \\ \lambda &\mapsto \gamma(1-\lambda) \end{aligned}$$

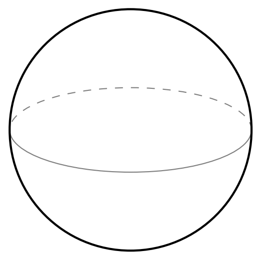
Hence, the fundamental group is indeed a group.

Examples.

(a) The 2-sphere is defined as the set:

$$S^2 := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

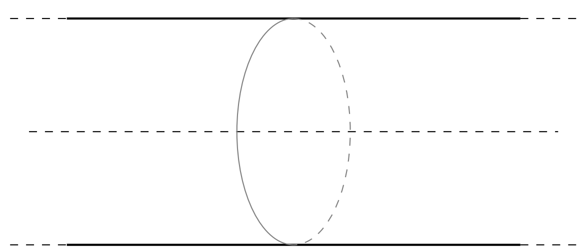
equipped with the subset topology inherited from $\mathbb{R}^3$.



A sphere has the property that all loops at any point are homotopic. Hence, the fundamental group (at every point) of the sphere is the trivial group

$$\forall p \in S^2 : \pi_1(p) = 1 := \{[\gamma_e]\}$$

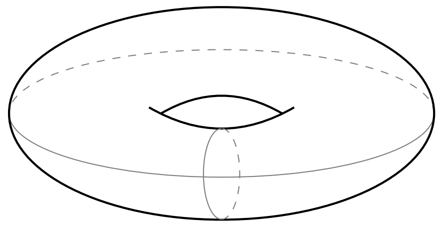
(b) Consider the cylinder $C := \mathbb{R} \times S^1$ equipped with the product topology.



A loop in C can either go around the cylinder (i.e. around its central axis) or not. If it does not, then it can be continuously deformed to a point (the identity loop). If it does, then it cannot be deformed to the identity loop (intuitively because the cylinder is infinitely long) and hence it is a homotopically different loop. The number of times a loop winds around the cylinder is called the *winding number*. Loops with different winding numbers are not homotopic. Moreover, loops with different *orientations* are also not homotopic (so we have negative numbers now) and hence we have

$$\forall p \in C : (\pi(p), \bullet) \cong_{\mathrm{grp}} (\mathbb{Z}, +)$$

(c) The 2-torus is defined as the set $T^2 := S^1 \times S^1$ equipped with the product topology.



A loop in T^2 can intuitively wind around the cylinder-like part of the torus as well as around the hole of the torus. That is, there are two independent winding numbers and hence:

$$\forall\, p \in T^2 : \pi_1(p) \cong_{\mathrm{grp}} \mathbb{Z} \times \mathbb{Z},$$

where $\mathbb{Z} \times \mathbb{Z}$ is understood as a group under pairwise addition.

As a reminder, no known collection of topological invariants exists such that verifying that the two topological spaces match on all invariants in the list implies that the two spaces are homeomorphic.

3 Topological Manifolds and Bundles