Dynamical Systems Notes

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1.1 What is a dynamical system?

Let X be a space, f a function, and a semigroupoid which is usually $\mathbb{N}, \mathbb{Z}, \mathbb{R}$, or $\mathbb{R}^+$. A dynamical system is a triple $(X, f, \mathbb{N})$ of these objects. In practice, the point of specifying this triple is to iterate a function. Letting $x \in X$ be some point, we might consider the sequence

$$x, f(x), \underbrace{f^2(x)}_{f(f(x))}, f^3(x), \ldots$$

The above is the case where we have the natural numbers. If we have a group instead, like $\mathbb{Z}$, we will have inverses $f^{-1}, f^{-2}, \ldots$ as well. The third parameter in our triple is usually called the **time parameter**. If this parameter is $\mathbb{R}$, then we can have invertible and non-invertible flows, which we will discuss later. Other names for the objects of this triple are **phase space** for X and **evolution function** for f.

For now, we restrict our discussion to the discrete case. For the discrete case, we can define an **orbit** through the point x, which will look like the set

$$\{x, f(x), \ldots, f^n(x)\}$$

This is a positive orbit. In the case where we have an invertible map, we may have backward iterations in the orbit, as below

$$\{\ldots, f^{-1}(x), x, f(x), \ldots\}$$

Orbits are a key part of dynamical systems.

Example. *Heron's calculation of square roots.*

A historical example of a dynamical system involves taking $X = \mathbb{R}$, $\mathbb{N}$ as the set for our time parameter, and our function as

$$f(x) = \frac{1}{2}\left(x + \frac{A}{x}\right)$$

The orbit through a positive x converges to the square root of A.

Proof. Fix a positive α. Let us denote $x_n := f^{n-1}(x)$ and choose $x_1 > \sqrt{\alpha}$. Observe that

$$x_2 = \frac{x_1^2 + \alpha}{2x_1} < \frac{x_1^2 + x_1^2}{2x_1} = x_1$$

Also

$$x_1^2 - 2x_1\sqrt{\alpha} + \alpha = (x_1 - \sqrt{\alpha})^2 > 0$$

By rearrangement

$$x_2 = \frac{1}{2}\left(x_1 + \frac{\alpha}{x_1}\right) > \sqrt{\alpha}$$

By induction, we see that $x_n > x_{n+1}$ and that $x_{n+1} > \sqrt{\alpha}$. This shows that sequence $(x_n)_{n\in\mathbb{N}}$ is monotonically decreasing and bounded below. So, by the monotone convergence theorem, this sequence converges. Further, $\lim_{n\to\infty} x_n = \sqrt{\alpha}$. To see this, note that $\lim x_n = \lim x_{n+1}$ so that

$$x = \frac{1}{2}\left(x + \frac{\alpha}{x}\right) \implies x = \sqrt{\alpha}$$

■

We begin with a more geometrical approach. First, we remark that X is not usually just a set, but it may have some structure. For instance, X could be a topological space, a space equipped with a measure, and so forth. In the same way, we may be talking about smooth functions, or continuous functions, or measurable functions.

Definition. *Fixed point, Periodic point*
Let X be a topological space. If $x \in X$ is a fixed point, then

$$x = f(x)$$

From this, we have $f^n(x) = x$ for all $n \in \mathbb{N}$.
A point with period n satisfies $f^n(x) = x$. We are usually interested in the minimal period of a point.

Fact. Points on the real line must be fixed, period 2, or be nonperiodic. However, in higher dimensions, or even in the circle, there are more options.

Proposition. The minimal period of a point is a divisor of any period of that point.

Proof. Let m be the minimal period of a point x and let n be any period. Observe that

$$x = f^n(x) = f^{km+r}(x) = f^r(x)$$

where k, r are integers furnished by the division algorithm and $0 \leqslant r < m$. But since m is minimal, we must have $r = 0$. Hence $m \mid n$. ■

Remark. There are two ways to think about a dynamical system: in a geometric manner and an algebraic manner. For instance, we may think of a dynamical system as a formula, or alternatively as a graph of a function. There are advantages to both approaches. The geometric approach may allow us to imagine a richer set of functions, while the algebraic approach yields specific properties of certain dynamics.

Examples. We cover some basic examples of dynamical systems. Let $X = \mathbb{R}$ and suppose we have a discrete time parameter.

1. f given by $x \mapsto x + \alpha$, this is a shift operation.
2. f given by $x \mapsto kx$, for $k \in \mathbb{R}$.

Now, consider $S^1 = \mathbb{R}/\mathbb{Z}$ to be our space. With this quotient space, we have the property that $f(x+1) = f(x)$.

1. f given by $\varphi \mapsto \varphi + \alpha$. In this case, the shift on the circle is considered a rotation, and may be denoted R_α
2. f given by $\varphi \mapsto 2\varphi$, or $\varphi \mapsto n\varphi$ more generally. This multiplication should always be by an integer n. We denote these maps $E_2, E_3, \ldots, E_n, \ldots$
   This is an example of a **circle covering**. Unlike in the real line, when our phase space is S^1, we can only have multiplication by integers.
   (a) To elaborate on why n must be an integer, it is to ensure that the function is well-defined. For instance, if $k \in \mathbb{Z}$, then $\varphi \sim \varphi + k$. Applying the function, we see
   $$f(\varphi) = n\varphi \sim n(\varphi + k) = f(\varphi + k)$$
   If n is not an integer, we may not have $n\varphi \sim (n\varphi + nk)$.
   (b) Observe that n specifies how many times the circle wraps around itself. To be explicit, if φ is a point on a circle and we have the map $\varphi \to n\varphi$, then the points
   $$\frac{\varphi}{n}, \frac{\varphi+1}{n}, \ldots, \frac{\varphi+(n-1)}{n}$$
   all map to φ.

Q: Given the map E_a, how many fixed points are there?
To approach this question, it is useful to think of a different base number system. Every point on the circle corresponds to some base representation. For instance, given E_{10}, the numbers correspond to the standard decimal system. So, suppose we represent our number as

$$x = 0.x_1x_2x_3\ldots, \quad x_i \in \{0, \ldots, a-1\}$$

We may write $x = \sum_{i=1}^{\infty} x_i a^{-i}$ as well. If our number is written in base a, then we see that $E_a(x) = 0.x_2x_3\ldots$ Now, to answer our question about fixed points, we know that if x is a fixed point, then $E_a(x) = x$. Hence, it follows that $x_1 = x_2$, $x_2 = x_3$, and so forth. Therefore, the only fixed points are those where all x_i are equal. This yields the fixed points

$0, 0.111\ldots, 0.222\ldots, \ldots, 0.(a-1)(a-1)\ldots$. However, the last point $0.(a-1)(a-1)\ldots = 1$, so there are only $a-1$ fixed points.

Q: Now, what about the number of periodic points of period n?

This means that $x_{i+n} = x_i$ for all i. So, we may choose digits from 0 to n, where we have a choices for each digits. There are a^n possible choices. Again, we exclude the case where we choose only $(a-1)$ for our digits (as this gives 1). Therefore, there are $a^n - 1$ periodic points of period n.

Q: Why is this interesting and why do we care about this algebraic representation?

Suppose we consider periodic points of E_a with period p. Consider the possible minimal periods of these points. If a point has period p, then by our earlier proposition its minimal period must be a divisor of p. Hence, this point has minimal period 1 (i.e. it is a fixed point) or p. As a result, the total number of points with minimal period p is

$$a^p - 1 - (a-1) = a^p - a$$

Consider the orbit

$$\{x, E_a(x), \ldots, E_a^{p-1}(x)\}$$

This shows us that points with period p occur in groups of p, which gives us Fermat's Little Theorem:

$$p \mid a^p - a$$

1.2 Topological Ideas in Dynamical Systems

We move to some topological ideas. The most generic idea is called an ω-limit set. Given the forward orbit

$$\{x, f(x), \ldots, f^n(x), \ldots\}$$

this is a sequence of points in X. The set of limit points of this sequence is called an ω-limit set. For instance, invoking Heron's statement from earlier is equivalent to saying

$$f(x) = \frac{1}{2}\left(x + \frac{A}{x}\right)$$
$$\omega(x, f) = \{\sqrt{A}\}$$

Since the ω-limit set depends on the orbit, it depends on both the point x and the map f. We may also specify an α-limit set, which consists of the limit points of the backward orbit.

Definition. *ω-limit and α-limit sets*
To be explicit, we may write

$$\omega(x, f) = \{y \in X : \exists \text{ a sequence } n_k \text{ such that } f^{n_k}(x) \to y\}$$
$$\alpha(x, f) = \{y \in X : \exists \text{ a sequence } n_k \text{ such that } f^{-n_k}(x) \to y\}$$

Let X be a sequentially compact topological space. Recall that a space X is sequentially compact if every sequence in X has a convergent subsequence whose limit is also in X. The sequential compactness of X implies that $\omega(x, f)$ is nonempty.
Clearly, if a space is not sequentially compact, it is possible that $\omega(x, f)$ is empty. Simply consider the shift operator $x \mapsto x + \alpha$ on $\mathbb{R}$, from which we may obtain the sequence $\{1, 2, 3, \ldots\}$. No subsequence of this sequence converges.
The situation differs if we consider the rotation operator on the circle. There are two different cases: rotation by a rational number and rotation by an irrational number.
Claim: In the case of rotation by a rational number, all points are periodic.
Let $\alpha = \frac{m}{q}$ and consider R_α. Observe that

$$R_\alpha^q = x + q \cdot \frac{m}{q} = x + m = x \pmod 1$$

Hence, the ω-limit set just consists of the orbit of x

$$\omega(x, f) = \{x, R_\alpha(x), \ldots, R_\alpha^{q-1}(x)\}$$

Claim: In the case of rotation by an irrational number, then for all x, $\text{orb}(x)$ is dense in the circle.
Remark. If your orbit is dense, then the ω-limit set is the entire phase space.
Exercise: Prove this claim.

Theorem. *Kronecker-Weyl. Equidistribution of rotation on a circle.*
Let $\alpha \notin \mathbb{Q}$, $x \in S$ and $I \subseteq S$, where S is the unit circle. We may define

$$A_n = \frac{\#\{m : 0 \leqslant m \leqslant n \text{ and } R_\alpha^m(x) \in I\}}{n+1}$$

This set A_n represents the frequency of our arrivals on the interval I. The theorem states that

$$\lim A_n \rightrightarrows |I|$$

where the double arrows indicate uniform convergence.

Informally, this theorem says that on any interval with length $\frac{1}{100}$, one out of every one hundred points will lie in this interval. We now consider applications of this theorem.

1.2.1 Applications of Kronecker-Weyl

First signs (leading digits) of powers of 2. The first few powers of 2 are $2, 4, 8, 16, 32, \ldots$ and the leading digits are $2, 4, 8, 1, 3, 1, 2, 5, 1 \ldots$. We are interested in the frequency of these leading digits. How often do 1s or 2s appear?
To begin addressing this question, we should observe that N starts with k if and only if

$$\log_{10} k \leqslant \{\log_{10} N\} < \log_{10}(k+1)$$

To show this, we may write

$$N = k.N_1 N_2 \ldots N_n \cdot 10^n$$

Clearly

$$\begin{aligned} k \cdot 10^n \leqslant k.N_1 N_2 \ldots N_n \cdot 10^n &< (k+1) \cdot 10^n \\ \log_{10} k + n \leqslant \log_{10}(k.N_1 N_2 \ldots N_n) + n &< \log_{10}(k+1) + n \\ \log_{10} k \leqslant \{\log_{10} N\} &< \log_{10}(k+1) \end{aligned}$$

We note that

$$\log_{10} N = \log_{10}(kN_1 N_2 \ldots N_n) = \log_{10}(k.N_1 N_2 \ldots N_n) + n = \{\log_{10}(N)\} + [\log_{10}(N)]$$

This establishes the forward direction. On the other hand suppose the inequality holds. We can work backwards from the equations above to get the conclusion. Now, we use this fact and consider numbers 2^n. If 2^n starts with k, then

$$\log k \leqslant \{n \log_{10} 2\} < \log(k+1)$$

Notice that taking the fractional part is the same as factorizing $\mathbb{R}$ over $\mathbb{Z}$. So we may reframe this problem as rotation on a circle through the angle $\log_{10} 2$. We want to know how often $R_{\log_{10} 2}$ falls on the interval $[\log k, \log(k+1)]$. Call $\alpha = \log_{10} 2$. We know that $\alpha \notin \mathbb{Q}$, otherwise we reach the contradiction $10^p = 2^q$. By the Kronecker-Weyl theorem, we conclude that

$$\begin{aligned} \frac{\#\{m : 0 \leqslant m \leqslant n, R_\alpha^m \in [\log_{10} k, \log_{10}(k+1)]\}}{n+1} &\rightrightarrows |\log(k+1) - \log k| \\ &= \log_{10}\left(\frac{k+1}{k}\right) \end{aligned}$$

Based on this formula, we see that 1 is the most common digit, 2 is the second most common digit, and since the function is decreasing, 9 is the rarest digit.
A quick review of what we did:

1. We found an equivalent inequality for leading digit of a number.
2. We recognize that rotation by $\log_{10} 2$ (recall that rotation means adding $\log_{10} 2$) corresponds to multiplying by 2.

3. We used this inequality, the irrationality of the angle, and $\mathbb{R}/\mathbb{Z}$ to leverage Kronecker-Weyl.

1.3 More on limit sets

Remark. The ω-limit set $\omega(x, f)$ is closed, since the set of subsequential limits of a sequence is closed.

Remark. For rotations, all ω-limit sets are isometric. For coverings, we have a entirely different space that depends on the point.

Recall the Cantor set, which consists of the intersection of intervals with successively removed middle thirds on $[0, 1]$. Equivalently, we can consider it the set of points, in ternary, which do not contain the digit 1 anywhere.

$$K = \left\{ \sum_{i=1}^{\infty} a_i 3^{-i} : a_i = 0 \text{ or } a_i = 2 \right\}$$

It is clear that $\omega(x, E_3) \subseteq K$ (this is just a left shift of the ternary representation). How can we prove that any point of the Cantor set is a limit point of the orbit of an irrational x?

Let $k \in K$. The key idea here is that all finite sequences of 0s and 2s will appear infinitely often in the representation of an irrational x. Hence, given some $\varepsilon > 0$, we can always find some $n \in \mathbb{N}$ such that $|E_3^n(x) - k| < \varepsilon$.

Now, we discuss maps related to a geometrical view.

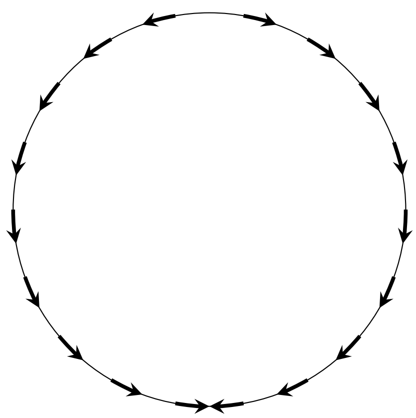
1. The North-South map. Given a circle, we imagine that the image of any point, except the North pole, goes to the South pole.

$$\omega(x, f) = \begin{cases} S, & \text{if } x \neq N \\ N, & \text{otherwise} \end{cases}$$

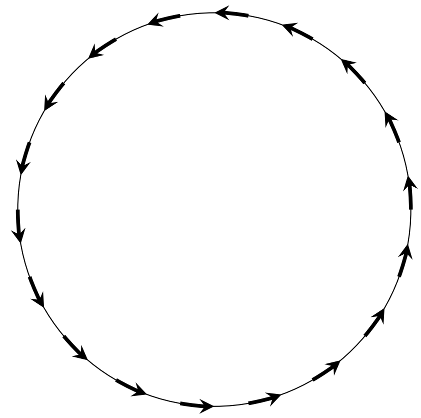
   Note that a North-South map may be more abstract in that only two fixed points (with the described dynamics) are required which do <u>not necessarily have to be at the North pole and South pole.</u>

2. The parabolic map. In this map, all points have the same limit, that is $\omega(x, f) = P$. Observe how this map has different dynamics from the North-South map. With the North-South map, a small neighborhood around S will go to S. In contrast, for the parabolic map, in every neighborhood of P, there are points that will go outside of the neighborhood when f is applied (though they will eventually return).

These are two intuitively helpful objects in 1D dynamics.



North–South map



Parabolic map

Example. Suppose f_1 and f_2 are both north-south maps where these maps have different sets of poles. As an exercise, determine if $f_2 \circ f_1$ is a (nontrivial) rotation.

In the case of a north-south map, if we consider a neighborhood U around the point S, then we see that

$$\overline{f(U)} \subseteq U$$

In other words, the region maps into itself. In this case, we call U an **absorbing domain.**

1.4 A General Perspective on Dynamical Systems

In the above lesson, we started with basic examples of dynamical systems involving functions such as the north-south map, rotations, doublings and triplings. More serious study of dynamical systems begins with spaces of functions. For instance, we might consider:

1. The set of homeomorphisms of a circle, $\text{Homeo}(S^1)$
2. The set of smooth maps of a circle $\mathscr{C}^1(S^1)$.
3. The set of diffeomorphisms of a circle, $\text{Diffeo}(S^1)$. A diffeomorphism is a smooth, invertible function whose inverse is also smooth.

All of these sets are in fact topological spaces. We may act in these spaces.
Usually, when we speak about dynamical systems, we focus on the function, f. However, the point x in the space may also be important. This is demonstrated in **Bernoulli shifts**. A Bernoulli shift acts on the set of sequences. We may consider their action on one-sided sequences first:

$$\sigma : [0,1]^{\mathbb{N}} \to [0,1]^{N}, \quad 010011\ldots \mapsto 10011\ldots$$

This shift looks exactly like doubling on the circle for binary numbers. We may define

$$\Sigma_+^2 = \{(x_n)_{n \geqslant 0} \mid x_n \in \{0, 1\}\}$$

The subscript + and superscript 2 indicate one-sided sequences and the number of choices we have for terms of the sequence, respectively. We may also define a metric d on Σ_+^2 by $d(x, y) := 2^{-n}$, where n is the first index where the sequences x and y differ. If they do not differ, then $d(x, y) = 0$.

1.5 Exercises

1. In the case of rotation by an irrational number, then for all x, $\mathrm{orb}(x)$ is dense in the circle.

   Proof. Suppose, for contradiction, that $\mathrm{orb}(x)$ is not dense in the circle. Hence, there exists $y \in S^1$ such that y is not a limit point of $\mathrm{orb}(x)$. This means that there exists $\varepsilon > 0$ such that $(y - \varepsilon, y + \varepsilon) \cap \mathrm{orb}(x) = \varnothing$. Call our irrational rotation α. Since α is irrational, we can find any finite sequence at some point in it. So, if $x < y$, we can shift x by a number that is arbitrarily close to $y - x$. If $y < x$, then we can shift it by some value arbitrarily close to $1 + y - x$. In either case, we have shown that $(y - \varepsilon, y + \varepsilon) \cap \mathrm{orb}(x)$ is nonempty, and we have reached a contradiction ($\lightning$).

   To be explicit, every time we rotate 10 times, it shifts α to the left. Since α is irrational, we can eventually find some finite sequence that matches the first N digits of $y - x$ to make it arbitrarily close.

2. Suppose f_1 and f_2 are both north-south maps where these maps have different sets of poles. As an exercise, determine if $f_2 \circ f_1$ is a (nontrivial) rotation.

   Solution. $f_2 \circ f_1$ will be a rotation. But it seems like it will depend on exactly where the four points lie on the circle. If we imagine four points on the circle and label them N_1, S_1, N_2, S_2, we can see that there will be four sections of the circle where points in these sections will rotate by different amounts.

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2.1 Proof of the Kronecker-Weyl Theorem

Recall the Kronecker-Weyl theorem. We will use the same notation in this theorem. In particular, recall that

$$A_n = \frac{\#\{j : 0 \leqslant j \leqslant n \text{ and } R_\alpha^j(x) \in I\}}{n+1}$$

The theorem says that A_n uniformly converges to the length of the interval under consideration, $|I|$. We will prove the theorem. Later, we will find that another perspective of this statement involves the concept of a uniquely ergodic map.

Proof. Consider the characteristic function of I

$$\chi(x) = \begin{cases} 1, & \text{if } x \in I \\ 0, & \text{otherwise} \end{cases}$$

Using this characteristic function, we may write

$$A_n = \frac{\sum_{j=0}^n \chi(\{x + \alpha j\})}{n+1}$$

We want to summarize this function. We start with trigonometric monomials. The point is the substitute this characteristic function with a trigonometric polynomial that is close enough. For now, we will assume that it is possible to decompose the characteristic function as the sum of trigonometric monomials of the form $e^{i\pi k\varphi}$. Observe that

$$\begin{aligned} \frac{\sum_{j=0}^n e^{i\pi k(x+\alpha j)}}{n+1} &= \frac{e^{i\pi kx}\sum_{j=0}^n e^{i\pi k\alpha j}}{n+1} \\ &= \frac{e^{i\pi kx}}{n+1} \cdot \frac{1 - e^{i\pi k\alpha(n+1)}}{1 - e^{i\pi k\alpha}} \end{aligned}$$

where we have used the formula of the geometric series to arrive at the last equality. Taking the modulus, we have

$$\left| \frac{e^{i\pi kx}}{n+1} \cdot \frac{1 - e^{i\pi k\alpha(n+1)}}{1 - e^{i\pi k\alpha}} \right| \leqslant \frac{2}{n+1} \cdot \frac{1}{|1 - e^{i\pi k\alpha}|} \xrightarrow[n\to\infty]{} 0$$

We note here the importance of the irrationality of α, for if α were rational, it is possible for $|1 - e^{i\pi k\alpha}|$ to be zero. The limit above is, in fact, uniform. Now, we define $P := c_0 + c_k e^{i\pi kx}$. Using what we know about the limit of trigonometric monomials above (multiplication by a constant and sums do not change this limit), we see that

$$\lim_{n\to\infty} A_n(P) = c_0 = \int_{S^1} P \, dx$$

It is possible to find two smooth functions f_1, f_2 on the circle satisfying $f_1 < \chi < f_2$. Further, we may choose these functions so that

$$\int_{S^1} (f_2 - f_1)\, dx < \varepsilon$$

for some $\varepsilon > 0$. We would like to approximate our functions f_1, f_2 with trigonometric polynomials P_1, P_2 so that $|f_1 - P_1| < \varepsilon$ and $|f_2 - P_2| < \varepsilon$. How do we find these polynomials? There are two strategies. The first is based on Weierstrass' theorem. The second is based on Fourier series. If we have $\mathscr{C}^2$ smooth functions, it can be shown that its Fourier coefficients are decreasing approximately at the rate $\frac{1}{n^2}$. This means that the remainder of the Fourier series is some function which may be estimated. The existence of these polynomials P_1, P_2 gives us

$$P_1 - \varepsilon < f_1 < \chi < f_2 < P_2 + \varepsilon$$

Taking sums, we have

$$A_n(P_1) - \varepsilon < A_n(f_1) < A_n(\chi) < A_n(f_2) < A_n(P_2) + \varepsilon$$

To be explicit, if F is a function, then

$$A_n(F) = \frac{\sum_{j=0}^{n} F(\{x + \alpha j\})}{n+1}$$

where we note that $A_n(F)$ is a function that takes in the argument x. Taking the limit of the previous inequality, we have

$$\int P_1 - \varepsilon \leqslant \lim A_n(\chi) \leqslant \int P_2 + \varepsilon$$

We observe one more fact. Since $P_2 - \varepsilon < f_2$, we have

$$\int P_2 - \varepsilon < \int f_2 < \int f_1 + \varepsilon < \int \chi + \varepsilon$$

As a result, $\int P_2 + \varepsilon \leqslant \int \chi + 3\varepsilon$. An analogous inequality holds when we consider $f_1 < P_1 + \varepsilon$, and putting these two inequalities together gives

$$\int \chi - 3\varepsilon \leqslant \lim A_n(\chi) \leqslant \int \chi + 3\varepsilon$$

Since ε was arbitrary, we may conclude $\lim A_n(\chi) = \int \chi$. Based on how we defined these functions, we see that

$$\lim_{n\to\infty} A_n = \lim A_n(\chi) = \int \chi = |I|$$

■

Aside. Elaboration on why P_1, P_2 exist.

1. *Stone-Weierstrass Theorem* (Weierstrass Approximation Theorem).
   If X is a compact Hausdorff space and $\mathcal{A} \subseteq \mathscr{C}(X)$ is a subalgebra containing the constants and separating points of X, then $\mathcal{A}$ is dense in $\mathscr{C}(X)$, with the uniform norm.
   On the circle, the algebra generated by $\{e^{ikx}\}$ separates points and contains constants, so by Stone-Weierstrass it is dense in continuous functions $\mathscr{C}(S^1)$. Thus, any continuous f on the circle can be approximated uniformly by a trigonometric polynomial.

2. *Approximation via Fourier series.*
   Any L^1 function f on the circle admits a Fourier series $f(x) \sim \sum_{k\in\mathbb{Z}} \hat{f}(k)e^{ikx}$.
   For piecewise $\mathscr{C}^1$ or $\mathscr{C}^2$ functions (and smooth functions), the partial sums of the Fourier series converge quickly to f. One can truncate the Fourier series and get a finite trigonometric polynomial that is close to f in the supremum norm or in L^1-norm.

2.2 Norms

1. Uniform norm (a.k.a. sup norm)

   For a continuous function f on a compact space X,

$$\|f\|_\infty = \sup_{x\in X} |f(x)|$$

   Saying that a sequence of functions f_n converges uniformly to f means $\|f_n - f\|_\infty \to 0$. Equivalently, $\sup_{x\in X} |f_n(x) - f(x)| \to 0$.

2. L^1 norm

   If f is (at least) integrable on a measure space (X, μ), then

$$\|f\|_1 = \int_X |f(x)|\, \mu(dx)$$

   Saying $\|f_n - f\|_1 \to 0$ means the average of $|f_n - f|$ over X goes to zero.

2.3 Flows

Flows are another type of dynamical system. A flow consists of a set of diffeomorphisms $\{g^t\}_{t\in\mathbb{R}}$ on some closed manifold. We assume the dependence on the parameter t is continuous.

A diffeomorphism is a smooth invertible map whose inverse is also smooth. The main property of a flow is given by

$$\boxed{g^{t+s}(x) = g^t(g^s(x))}$$

We also have that g^0 is the identity map. For now, let us consider our flows on the real line $\mathbb{R}$. We define

$$g^t(x_0) = x_0 \cdot e^t, \quad g^{t+s}(x_0) = x_0 e^t e^s = g^t(g^s(x_0))$$

Why did this work? Note that this is not a random function, but it is a solution to a differential equation, $\dot{x} = x$ with the initial condition $x(0) = x_0$. To investigate this situation more, let us begin with the closed manifold M and smooth vector field ν on M. Given this setup, consider the autonomous differential equation

$$\dot{x} = \nu(x(t))$$

where our initial conditions belong to some manifold. Suppose that there exists a solution to this differential equation $x(t, x_0)$, where x_0 indicates the initial condition. Now, we define a flow

$$g^t(x_0) = x(t, x_0)$$

See our previous example to better understand the meaning of this notation. How do we show that this has the required property? Observe that

$$g^{t+s}(x_0) = x(t+s, x_0) = x(t, x(s, x_0)) = g^t \circ g^s(x_0)$$

we should have uniqueness of solutions for this to hold (our setting above does yield this uniqueness). To be more formal, to show that $x(t+s, x_0) = x(t, x(s, x_0))$, observe that

$$\frac{d}{dt} x(t+s, x_0) = v(x(t+s, x_0))$$

So $x(t+s, x_0)$ satisfies the same differential equation, also, both differential equations have the same initial condition. For one equation, when $t = 0$, we have $x(0+s, x_0) = x(s, x_0)$, for the other equation, when $t = 0$, we have $x(0, x(s, x_0))$, which means that $x(0) = x(s, x_0)$ in both cases. Hence, our desired equality follows by the uniqueness of solutions.

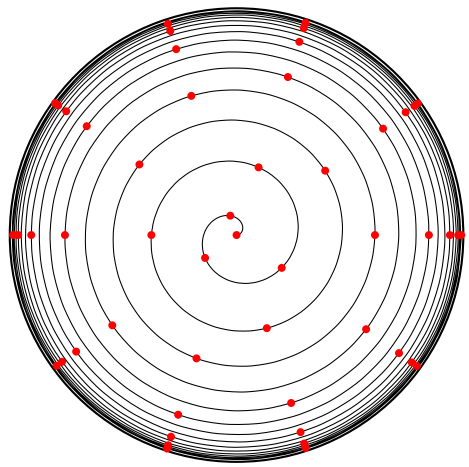
We briefly elaborate on some terminology. We have orbits and ω-limit sets in the discrete case. For flows, orbits are called **trajectories** and sometimes called solutions (i.e. of differential equations). Further, periodic orbits are called **cycles** in the context of flows. If we have $g^t(x) = x$ for all $t \in \mathbb{R}$, then x is called a **singular point**. If there exists τ such that $g^{t+\tau}(x) = g^t(x)$, then we have a cycle, then τ is called the **period** of the cycle. The full cycle itself is the set

$$\{g^t(x) : 0 \leqslant t < \tau\}$$

which is a closed loop in the manifold M. There is also an analogous concept to the ω-limit set, where given the **one-parameter family of functions** $\{g^t(x)\}_{t\in\mathbb{R}}$, we consider the set of all sequences of functions

$$\{g^{t_k}(x)\}_{k\in\mathbb{N}} \text{ such that } \lim t_k \xrightarrow{k\to\infty} \infty$$

The ω-limit set $\omega(x,g)$ is the set of all subsequential limits of these sequences of functions.
Example. As an example, we may consider an outward spiral (a trajectory) that gets closer and closer to the boundary of a circle (a cycle in the terms of a flow) on the inside.



By looking at the red points, we can see subsequences of functions that converge towards a point on the edge of the circle. Hence, $\omega(x,g)$ in this example is equal to the circle itself.

2.4 Properties of ω-limit Sets

We list some properties of the ω-limit set.

1. If we start with a closed manifold, then $\omega(x,g)$ is nonempty. Recall that an infinite set of points in a compact space must contain a limit point in that space.

2. Further, $\omega(x,g)$, being the set of subsequential limits, is itself closed.

3. $\omega(x,g)$ is invariant under the flow, that is

$$g^t(w(x,g)) = w(x,g)$$

   This is because applying g^t just means a renumbering of the sequence (i.e. just changing starting point of time).

4. The set $\omega(x,g)$ is connected.
   To see this, recall that by assumption g^t is continuous on $\mathbb{R}$. Since $\mathbb{R}$ is connected, the trajectory $\{g^t(x)\}_{t\in\mathbb{R}}$ is also connected in the phase space. There is a topological property that states if you take the set of all limit points of a connected space, it must also be connected. Hence, $\omega(x,g)$ is connected.

The first three properties hold for the discrete case, while the last property holds only in the case of a flow.

2.5 Flows on Surfaces

Let us first consider irrational winding on a torus. We can think of the torus as a rectangle with its opposite sides identified, or $\mathbb{R}^2/\mathbb{Z}^2$ or $[0,1] \times [0,1]$. Let $\mathbf{v} = (a, b)$, where a, b are numbers. Given the system of differential equations

$$\begin{cases} \dot{x} = a \\ \dot{y} = b \end{cases}$$

The solution is a pair of lines

$$\begin{cases} x(t) = \{at + x_0\} \\ y(t) = \{bt + y_0\} \end{cases}$$

On the $[0,1] \times [0,1]$ square, we start at point (x_0, y_0) and we travel in a line with slope $\frac{b}{a}$. Given the setup above, we have two cases. Either $\frac{a}{b} \in \mathbb{Q}$ or $\frac{a}{b} \notin \mathbb{Q}$. If $\frac{a}{b} \in \mathbb{Q}$, then all trajectories will be cycles. This corresponds to the idea that for circular rotation there are only periodic orbits.

To aid our analysis of the situation, we will define a helpful object. This is a very general object, and it has four names for each of its different contexts.

1. First return map
2. Poincaré map
3. Holonomy map (in complex analysis).
4. Monodromy (in general topology).

To apply this concept to our investigation of the torus, we can simply choose the real line L (intersecting with the bottom, horizontal edge of the square). The main point here is that the object is a hypersurface of the manifold. In our case, we have a 2-dimensional manifold, so the hypersurface is of dimension 1. Generally, for an n-dimensional space, the hypersurface will have dimension $n-1$. Now, we need that all vectors should not be tangent to the hypersurface. In this case, the hypersurface is called a **transversal hypersurface**.

We define a map, starting with an initial point $(\tilde{x}, \tilde{y})$, and a flow map

$$\varphi(t) = (x(t, x_0), y(t, y_0))$$

The first return map, P, maps $(\tilde{x}, \tilde{y})$ to the first time it returns to our hypersurface L.

$$P(\tilde{x}, \tilde{y}) = \varphi(s), \quad s = \min\{t > 0, \varphi(s) \in L\}$$

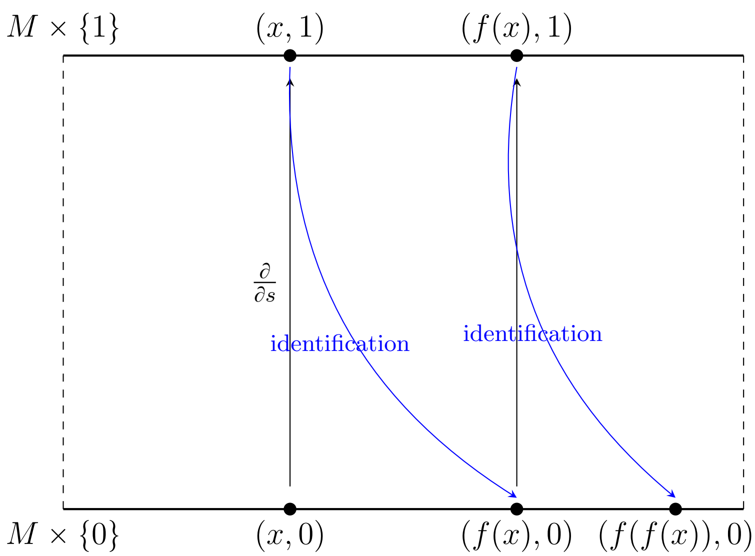
If we take (x_0, y_0) to be a point on the torus, we can find the point of first return by solving $y(t) = 0 \pmod 1$ for the minimal $t > 0$ and plugging this into $x(t)$. Observe that $x \mapsto \{x + \frac{b}{a}\}$ is a rotation. If we have irrational rotation, we have a dense orbit for every point. In other words, $\omega(x, R_\alpha) = \mathbb{T}^2$ for $\alpha \notin \mathbb{Q}$. In contrast, the orbit of a rational rotation will not be dense.
In turns out that there exist no smooth flows on a sphere (or any other shape) with ω-limit sets that are equal to the entire shape.

2.6 Dual to Poincaré Map

The dual object to the Poincaré map is called the Smale construction. This is also called a suspension or mapping torus. The idea is

$$f \xrightarrow{\text{Smale}} g^t, \quad f \xleftarrow{\text{Poincaré}} g^t$$

where the setting of g^t is N, an $n+1$-manifold, and the setting of f is M, an n-dimensional manifold.



We take the following steps to construct a flow from a map

1. Start with a map $f : M \to M$. We may think of f as the first-return map restricted to a cross-section of some flow (i.e. restricting the map P to L in the previous example)
2. Build the $(n+1)$-manifold N by taking the product $M \times [0, 1]$ and identifying

$$(x, 1) \sim (f(x), 0)$$

   In other words,

$$N = M \times [0,1] \big/ (x,1) \sim (f(x),0)$$

This quotient N is often called the **mapping torus** of f. We may consider a point (x, s), where $x \in M$ and $s \in [0, 1]$. After identification, this point will be a representative for an equivalence class.

3. Get a flow on N by pushing forward the trivial vertical flow on $M \times [0, 1]$. Concretely, on the cylinder $M \times [0, 1]$, we may consider the the vector field $\frac{\partial}{\partial s}$ (where s is the $[0, 1]$ coordinate). Then, because of the identification $(x, 1) \sim (f(x), 0)$, the field "glues up" to a well-defined flow on N.
   - The notation $(0, \frac{\partial}{\partial s})$ with $v = 0$ is the vertical vector field on $M \times [0, 1]$. We push it forward onto the quotient N. At a point (x, s), the vector is solely in the $[0, 1]$ direction, i.e. the constant unit vector $(0, \ldots, 0, 1)$.
4. Relate the flow to the Poincaré map. In the new flow on N, we can take the cross section, where $s = 0$. The Poincaré (or first-return) map of the flow on that cross section is precisely f.

2.7 Exercises

1. Consider a torus in $\mathbb{R}^3$. If we have a rational angle of rotation, then the orbit will be closed, and hence a knot. What is the rational number needed to obtain a trefoil on the torus?

   Solution. The angle of rotation should be 120 degrees.

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Recall that we discussed the Smale construction as a way to find a corresponding flow given some map. Suppose we want to find a correspondence between flows in dimension n with maps in dimension n. A simple way to obtain a map from a flow $\{g^t\}$ is to say $f = g^1$. Moreover, there is a specific and important case for diffeomorphisms. Of course, not all diffeomorphisms have this form, that is, we cannot find a corresponding flow to any diffeomorphism. Why?

Proposition. Take a flow $\{g^t\}$ and set $f = g^1$. All periodic points p are fixed or non-isolated.

The main idea of the proof is that we either have a fixed point of a vector field, or a periodic cycle of the vector field.

Proof. Consider the map g^1 and assume we have a periodic point satisfying $g^n(x) = x$. By the property of a flow, we have that for all τ

$$g^n(g^\tau(x)) = g^{n+\tau}(x) = g^\tau(g^n(x)) = g^\tau(x)$$

Hence, for all τ, we see that $g^\tau(x)$ is a periodic point of g. Another way to say this is that all periodic points come from a closed orbit $\{g^\tau(x) : \tau \in \mathbb{R}\}$. We have a map $\tau \mapsto g^\tau(x)$ from $\mathbb{R}$ to our manifold M. We note that this is a continuous map (by assumption, since g is a flow). Since this map is continuous and $\mathbb{R}$ is connected, the image is a connected set. Now, there are only two options. If there is more than one point in the image, then these points are not isolated because the image is connected. If there is only one point in the set, then it means

$$g^\tau(x) \equiv x, \ \forall \tau$$

In particular, we have $g^1(x) = x$, meaning x is a fixed point.

■

This proposition shows that there are diffeomorphisms which are not realizable as a time-one map of a continuous-time flow. If we have a diffeomorphism whose periodic points are a finite cycle, then by the proposition, it is not a diffeomorphism of a continuous-time flow. An example is given in Lecture 6.

3.1 The Poincaré-Bendixson Theorem

Our next goal is to discuss flows in small dimension. For this purpose, we must discuss the Poincaré-Bendixson theorem.

Definition. *Polycycle*
We start with a finite set of fixed points $S_1, S_2, \ldots, S_n$. There are trajectories which may connect this set of fixed points. It is useful to think of these as fixed points as nodes and arrows between fixed points as directed edges. For instance, if we have x on the trajectory and between fixed points S_1 and S_2, then $\omega(x) = S_2$ and $\alpha(x) = S_1$.

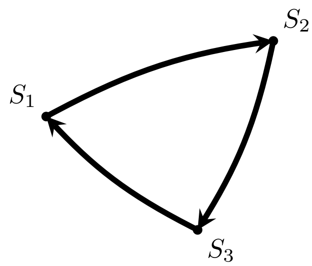


Figure 1: A polycycle

Theorem. *Poincaré-Bendixson Theorem*
Let $\{g^t\}$ be a flow on S^2, the sphere, with a finite number of fixed points. We mean that there is a finite number of points satisfying $g^t(x) = x$ for all t. Then, the ω-limit set of x must be a

1. fixed point,
2. cycle,
3. or polycycle.

Proof. Assume that there are no fixed points in $\omega(x)$. Take any point $y \in \omega(x)$ and consider $\omega(y)$. Choose $z \in \omega(y)$. Since ω-limit sets are invariant, we must have $g^t(y) \in \omega(x)$. Since

$$z = \lim_{k \to \infty} g^{t_k}(y) \in \omega(x)$$

It follows that $z \in \omega(x)$ by the property that ω-limit sets are closed. Define $y_k := g^{t_k}(y)$. By our hypothesis, there are no singular (fixed) points in $\omega(x)$, so z must change as time changes (it is moving on some trajectory). Since g^t is a flow and we are dealing with continuous maps, the direction of the trajectory through points near z must smoothly change. In particular, this applies for points y_k and y_{k+1} which are close to z. Recall that since $y_k, y_{k+1} \in \omega(x)$, these are limit points of the trajectory of x. The key point now is to consider a boundary curve Γ which consists of the portion of the trajectory between y_k and y_{k+1} and connecting the point y_k back to y_{k+1}. We label the interior of this boundary curve orange (call it R) and the exterior white and we are able to make this separation of regions only because we are on the sphere (i.e. in the case of this same boundary curve on a torus, it will not always split the torus into two regions). There are two cases now depending on whether the flow vectors along Γ point inward or outward. If they point inward, then orbits inside the boundary remain there forever (forward-invariance). If they point outward, then if one starts inside, then one stays inside if one goes backward in time (backward-invariance). We are able to arrive at a contradiction in both cases (how? something involving y_1 not being in the region), from which we conclude that $\omega(x)$ must be a cycle.

- We cannot have an interior point y_1 of R (the orange region) be approached by the orbit of x unless x were already inside R from the start. If x is inside, then the entire ω-limit set would lie in R, but this contradicts the assumption that $\partial R \subseteq \omega(x)$ is also approached from the outside.

Q: We need to revisit why such a drawing makes sense given the setup. The idea of a spiral inward containing points of $\omega(x)$ is weird.

On the other hand, suppose that there are only fixed points in $\omega(x)$. There can only be one fixed point in $\omega(x)$ since the set is connected and finite.

Finally, suppose that there are fixed points and non-fixed points in $\omega(x)$. Again, we choose some $y \in \omega(x)$ and some $z \in \omega(y)$. There are two cases for z, either it is a fixed point or not.

- If z is not a fixed point, then we are completely in the first case, and in fact there are no fixed points in $\omega(x)$.

- If z is a fixed point, then $\omega(y)$ consists only of the fixed point z. So the trajectory $\{g^t(y) : t \in \mathbb{R}\} \subseteq \omega(x)$ and goes to one fixed point z. Now, the same situation holds for any $y \in \omega(x)$. Since we know there are a finite number of fixed points, the trajectory of y must go to one of these fixed points. Therefore, we have a polycycle.

■

Remark. On a torus, we have either irrational winding, or something similar to Poincaré-Bendixson on a sphere.

Remark. Understanding the ω-limit sets is important for understanding the dynamics of that map. Further, the understanding of fixed points in particular is important. In fact, dynamics near fixed points may be used to describe the behavior of all the points to some extent.

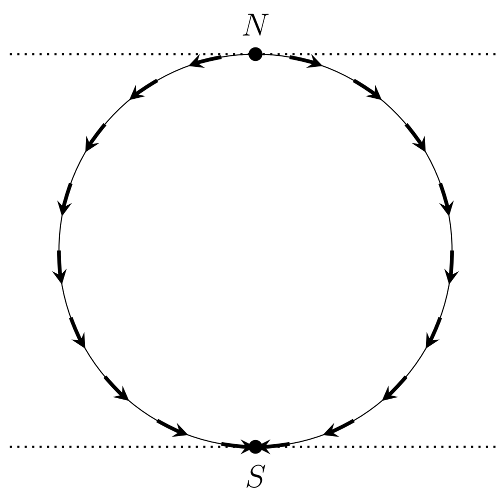
Let M be a smooth, compact, and closed manifold and let f be a diffeomorphism of M (this means $f \in \mathscr{C}^1(M), f^{-1} \in \mathscr{C}^1(M)$). Let x be a point of period q, that is $f^q(x) = x$. Consider the map

$$D_x f^q : T_x(M) \circlearrowleft$$

There are eigenvalues of $D_x f^q$, $\text{Eig}(D_x f^q)$, which is called the **spectrum** of the differential in the point, and these points individually are called **multiplicators** or **multipliers** of x. If we have

$$|\operatorname{Eig}(D_x f^q)| \neq 1$$

(i.e. in a discrete-time setting, none of the eigenvalues lie on the unit circle), then we can describe the dynamics using only this Jacobian matrix. This condition may also be used to describe a **hyperbolic point**. A key idea is that given the fixed points and the dynamics near the fixed point, we should be able to reconstruct the entire trajectory. We demonstrate this idea via the North-South map.



North–South map

Calling the North-South map f, we see that $|f'(N)| > 1$ and $|f'(S)| < 1$. The intuition here is that points near N move away from it, leading to an expansion, while points near S move towards it, leading to a contraction. The idea is that given two fixed points, one with a derivative larger than 1 and one with a derivative smaller than 1 we can restore the North-South map. Q: What is a more rigorous explanation for why we can relate the notion of 'expanding' to $|f'(N)| > 1$?

3.2 Hyperbolic points of S^2-flows

Because we are talking only about one point, the manifold itself is not important (i.e. this analysis is independent of whether we are on a sphere or torus). We will have a matrix and two eigenvectors in the two-dimensional case. Suppose our point is periodic. If our point is hyperbolic, then it is isolated and it is a fixed point (i.e. around a hyperbolic point, there is a neighborhood where it is the only fixed point). Suppose that our two eigenvalues are $\{\lambda, \mu\}$ for g^1 (we use g^1 for concreteness, but this argument will work for the other maps in the flow). There are two cases.

1. Both the eigenvalues are greater than one, or both are not.

$$|\lambda| > 1 \text{ and } |\mu| > 1$$
$$|\lambda| < 1 \text{ and } |\mu| < 1$$

   Now, regardless of the combinations above, we will have different pictures of the dynamics depending on the eigenvalues only.

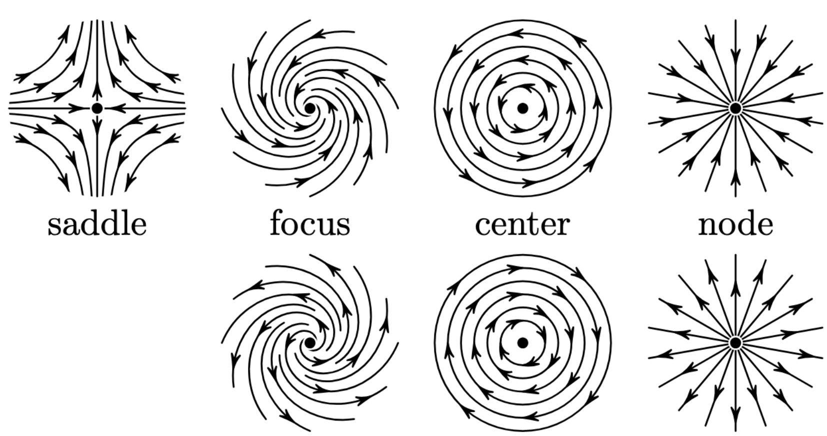
   (a) If $\lambda, \mu \in \mathbb{R}$, then the structure of the trajectory is given by knots, following the equation $y = x^{\frac{\lambda}{\mu}}$. The direction depends on the eigenvalues themselves. If both eigenvalues are greater than 1, then we go away from the origin and vice versa. The trajectories have the form of parabolas and are called **knots** in this case.

   (b) If $\lambda, \mu \notin \mathbb{R}$, then they will have the form $a \pm bi$. In this case, the value of a is most important. If $a < 1$, then trajectories go to the origin and if $a > 1$, then trajectories go out. The trajectory has the form of a spiral and is called a **focus**.

2. One of the eigenvalues is greater than one while the other is not.

$$|\lambda| > 1 \text{ and } |\mu| < 1$$
$$|\lambda| < 1 \text{ and } |\mu| > 1$$

   In this case, the trajectories have the form of hyperbolas and are called **saddles**.



Q: Why is the flow given by a system of differential equations near the hyperbolic point?

We introduced the saddle loop, with one hyperbolic point, and Bowen's eye, with two hyperbolic points, as two examples of hyperbolic polycycles.

3.3 Exercise

1. Let f be a flow with eigenvalues λ and μ at the hyperbolic point s. Given that $\omega(x) = \Gamma$, where Γ is a saddle loop, determine some equivalent conditions $F(\lambda, \mu) = 0$.

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4.1 Contraction Principle

The contraction principle is a very useful idea; it is often used in proofs for the existence and uniqueness of ordinary differential equations.

Definition. *Contraction*
We say that $f : X \to X$ is a *contraction* if there exists $\lambda < 1$ such that for all $x, y \in X$, if $x \neq y$, then $d(f(x), f(y)) < \lambda d(x, y)$, where d is a metric.
We will always assume that f is continuous and that X is a complete space.

Examples.

1. The function $x \mapsto \frac{x}{2}$ is a contraction.
2. Depending on interval on which it is defined, a function may or may not be a contraction. For instance, consider $x \mapsto x^3$, which is a contraction on $[-\frac{1}{2}, \frac{1}{2}]$, but not $[-1, 1]$.

Proposition. A contraction on a complete set has a unique fixed point a.

Proof. Consider the sequence $\{x, f(x), \ldots, f^n(x), \ldots\}$. Applying the definition of contraction inductively, we see that

$$\begin{aligned} d(f^n(x), f^{n+m}(x)) &\leqslant \sum_{i=0}^{m-1} d(f^{i+n}(x), f^{i+n+1}(x)) \\ &\leqslant \sum_{i=0}^{m-1} \lambda^{i+n} \underbrace{d(x, f(x))}_{C} \\ &\leqslant \frac{C\lambda^n}{1-\lambda} \xrightarrow{n\to\infty} 0 \end{aligned}$$

We have applied the triangle inequality m times in the first inequality $d(x, z) \leqslant d(x, y) + d(y, z)$. After, we applied the definition of a contraction. Finally, we used the equation for a geometric series (which is valid since $\lambda < 1$).
From the chain of inequalities, we see that this is a Cauchy sequence. Since the space is complete, this sequence must have a unique limit point. Therefore, we may conclude that $\omega(x)$ only contains one point a and that $f(a) = a$ from the invariance of the ω-limit set. This shows the existence of a limit point a.
To show uniqueness, assume that $f(b) = b$. We must have $d(a, b) = d(f(b), f(a)) \leqslant \lambda d(a, b)$ But this is only possible if $d(a, b) = 0$ and so $a = b$. ■

Fact. We state another useful application of contractions. Suppose we have a ball B and $f : B \to B$ and $\|Df\| < \lambda < 1$. Then, f is a contraction of B.

4.2 Topological Conjugacy

We need a more robust understanding of the topological spaces to investigate dynamical systems. Dynamical systems do not usually investigate maps alone, but the investigation usually focuses around neighborhoods or regions, and global properties which persist under small perturbation of our map. We begin with the idea of topological conjugacy.

Definition. *Topological Conjugacy*
Let X, Y be two topological spaces (usually manifolds) and $f : X \to X$ and $g : Y \to Y$. We call f, g topologically conjugated if there exists a homeomorphism $h : Y \to X$ which satisfies

$$f = h \circ g \circ h^{-1}$$

Remark. Smooth conjugacy is just a change of coordinates. Because we describe something similar to a change of coordinates, we should expect that many properties persist under topological conjugacy.

Examples.

1. For instance, suppose that x is a periodic point of g, so that $x = g^n(x)$. In this case, $h(x)$ will be periodic, with the same period, for f. To see this, Observe that

$$f^2 = (h \circ g \circ h^{-1}) \circ (h \circ g \circ h^{-1}) = h \circ g^{-2} \circ h^{-1}$$

We may argue inductively that $f^n = h \circ g^n \circ h^{-1}$. Then, it is clear that

$$f^n(h(x)) = (h \circ g^n \circ h^{-1})(h(x)) = h \circ g^n(x) = h(x)$$

2. This concept of conjugacy also helps us understand some intuitive facts. For instance, it helps to prove that rational rotations cannot be conjugate to an irrational rotation.

3. Consider the two functions $f(x) = \frac{x}{2}$ and $g(x) = \frac{x}{8}$. Is it possible for these functions to be conjugated?
Solution. First, conjugacy must send a periodic orbit to a periodic orbit. Hence, if such a function h exists, it must send 0 to 0. If we consider the derivatives, by the chain rule we have

$$f'(0) = (h \circ g \circ h^{-1})'(0) = h'(0) \cdot g'(0) \cdot \frac{1}{h'(0)} = g'(0)$$

Although it is not possible for a smooth map to be the conjugating map, if we take $h(x) = x^3$, we see that

$$h \circ f \circ h^{-1}(x) = \left(\frac{\sqrt[3]{x}}{2}\right)^3 = \frac{x}{8}$$

Note that this function is a homeomorphism but not a diffeomorphism since the inverse map $\sqrt[3]{x}$ does not have a derivative at 0.

4.3 More on Topological Spaces

Let M be a closed manifold. We also consider the spaces $\mathscr{C}^1(M)$ and Diffeo(M). We have metrics associated to each of these spaces

$$\mathscr{C}^1 : \rho_{\mathscr{C}^1}(f, g) = \|f - g\| = \sup_x \rho(f(x), g(x)) + \sup_x \|D_f(x) - D_g(x)\|$$
$$\rho_{\text{Diff}}(f, g) = \rho_{\mathscr{C}^1}(f, g) + \rho_{\mathscr{C}^1}(f^{-1}, g^{-1})$$

We are using ρ to indicate distance functions (metrics). We may generalize to spaces like $\mathscr{C}^2$ by using the Hessian map. With these metrics, we may now state the meaning of a neighborhood of a diffeomorphism. Recall the conditions of a hyperbolic point: given a periodic point $f^q(x) = x$, we have a differential map and its spectrum $|\text{sp}(D_{f^q}(x))| \neq 1$.

Proposition. *Persistence of Periodic Hyperbolic Points.*
Let x be a hyperbolic point with period q for a map f. Then there exists a neighborhood $V(x) \subseteq M$ and a neighborhood $U(f) \subseteq \text{Diff}(M)$ such that for all $g \in U(f)$, there exists a unique hyperbolic point $y \in V(x)$ with the same period q.

The point $y \in V(x)$ is usually called the **continuation** of the point x, for the map g.
Proof. With the same notations in the proposition, we define a *residue* $H = f^q - g^q$ and another function $F = f^q - \text{Id}$. Our goal is to show the existence of y. Towards this goal, we intend to solve the equation $g^q(y) = y$ for some point around a neighborhood of x. The idea is that if this point y is close enough to x, its multipliers (eigenvalues) will also be close (and not equal to 1), making y a hyperbolic point as well. Consider that

$$g^q(y) = y \iff f^q(y) - H(y) = y \iff F(y) + y - H(y) = y \iff F(y) = H(y)$$

From the premises, since x is a hyperbolic point with period q for a map f, it follows that $|\text{sp}(Df^q(x))| \neq 1$. Since $F = f^q - \text{Id}$, we may just subtract 1 from all the eigenvalues of Df^q to get the eigenvalues for DF and we see that $|\text{sp}(DF(x))| \neq 0$. So, DF is invertible. Hence, by the inverse function theorem, there exists an inverse map F^{-1} in a small neighborhood around x. Applying this inverse function, we now arrive at the equation $y = F^{-1}(H(y))$. Although we have no control over the eigenvalues of F^{-1}, we may choose g so that the eigenvalues of H should be less than ε. In particular, we may choose

$$\left\|DF^{-1}H\right\| \leqslant \underbrace{\left\|DF^{-1}\right\|}_{C} \cdot \|DH\| = C \cdot \|D_x(f^q - g^q)\| < C\varepsilon < \frac{1}{2} < 1$$

This is because the map H is close to 0 in the space of $\mathscr{C}^1$ functions. From the above, we see that $F^{-1}H$ is a contraction, and since a contraction on a complete set has a unique fixed point, we conclude that $F^{-1}H(y) = y$ exists.

■

Aside. We give an elaboration on some points in the above proof.

1. Why do the eigenvalues of DF come from subtracting 1 from those of Df^q?
   Observe that
   $$DF = D(f^q - \text{Id}) = Df^q - I$$
   Suppose that v is an eigenvalue of Df^q, so that $Df^q(v) = \lambda v$. We have
   $$DF(v) = Df^q(v) - Iv = (\lambda - 1)v$$

2. How do we justify each step in the last chain of inequalities?
   The non-obvious inequality is the first one. The fact that there exists a constant $C > 0$ such that $\|DF^{-1}\| < C$ is from a result called the Bounded Inverse Theorem.

3. Why is $F^{-1}H$ a contraction?
   Let F be a map on a convex, finite-dimensional Banach space X. If there exists a constant $\lambda < 1$ such that
   $$\|DF(x)\| \leqslant \lambda \text{ for all } x \in X$$
   then
   $$\|F(x) - F(y)\| \leqslant \lambda \|x - y\| \text{ for all } x, y \in X$$

We prove the last statement in the aside about the sufficient condition for a contraction.

Proof. We need the multi-dimensional mean-value theorem, sometimes called the mean-value inequality. Let $F : \mathbb{R}^n \to \mathbb{R}^n$. If $U \subseteq \mathbb{R}^n$ is convex, then for any $x, y \in U$, there is a continuous path $\gamma : [0, 1] \to U$ with $\gamma(0) = y$ and $\gamma(1) = x$ given by

$$\gamma(t) = y + t(x - y)$$

Then, by the fundamental theorem of calculus (in each component)

$$F(x) - F(y) = \int_0^1 DF(\gamma(t))(x - y)\, dt$$

Finally, with our premise, we see that

$$\begin{aligned}\|F(x) - F(y)\| &= \left\| \int_0^1 DF(\gamma(t))(x-y)\,dt \right\| \leqslant \int_0^1 \|DF(\gamma(t))(x-y)\|\ dt \\ &\leqslant \int_0^1 \|DF(\gamma(t))\| \cdot \|x-y\|\ dt = \|x-y\| \int_0^1 \|DF(\gamma(t))\|\ dt \\ &\leqslant \lambda \|x-y\|\end{aligned}$$

■

4.4 Structural Stability

Suppose we have M and some class of functions $\mathscr{C}$ on M. This class could be things like diffeomorphisms, diffeomorphisms with fixed points, $\mathscr{C}^1$ functions, and so on.

Definition. *Structurally Stable*
We say that $f \in \mathscr{C}$ is structurally stable in $\mathscr{C}$ if and only if there exists a neighborhood $U(f) \subseteq \mathscr{C}$ such that g is topologically conjugated to f for all $g \in U(f)$.

Intuitively, this means that if we perturb our map slightly, the properties of our map are preserved.

Examples.

1. We know that rotations are not structurally stable. To show this, note that we should have persistence of periodic orbits under topological conjugation. However, given a rotation by a rational number, we can always find a rotation nearby that is rotation by an irrational number.

2. The North-South map is structurally stable.

We prove that the North-South map is structurally stable.
Proof Sketch. Let f be the North-South map. First, we assume that $f'(N) > 1$ and $f'(S) < 1$ (i.e. the fixed points are hyperbolic). We proceed in two steps

1. In a small neighborhood $U(f)$ around f, all maps are of this type (i.e. they have the same dynamics with two fixed points).

2. North-South maps are topologically conjugated to each other.

To begin, given any North-South map, we know that given a small enough neighborhood around the North or South point, there is another map $\tilde{f}$ that has only one hyperbolic point in these two neighborhoods. Hence, we have $x - f(x) > \varepsilon$ and $x - \tilde{f}(x) > \frac{\varepsilon}{2}$ (using the fact that all points 'go down' towards the South pole and using the triangle inequality). Q: Why did we care about making $x - \tilde{f}(x) > \frac{\varepsilon}{2}$? For instance, since we are choosing $\tilde{f}$ arbitrarily close to f, we have $|\tilde{f}(x) - f(x)| < \delta$ for some $\frac{\varepsilon}{2} > \delta > 0$, then

$$x - \tilde{f}(x) = (x - f(x)) - (\tilde{f}(x) - f(x)) \geqslant (x - f(x)) - |\tilde{f}(x) - f(x)| > \varepsilon - \delta > \frac{\varepsilon}{2}.$$

So, we have a map $\tilde{f}$ with two fixed points $\tilde{N}$ and $\tilde{S}$, and we want to show conjugacy between this map and f. To do this, we start with the a region called the fundamental domain: it is the path between x and $f(x)$, denoted $[x, f(x)]$. We have another fundamental domain for $\tilde{f}$ denoted $[\tilde{x}, \tilde{f}(\tilde{x})]$, where we have chosen $\tilde{x}$ to not be a fixed point. Let $h : [x, f(x)] \to [\tilde{x}, \tilde{f}(\tilde{x})]$ be a homeomorphism. We extend this mapping h to $[f(x), f^2(x)] \to [\tilde{f}(\tilde{x}), \tilde{f}^2(\tilde{x})]$. To do this, pick $x \in [f(x), f^2(x)]$ and consider the extension $\tilde{f} \circ h \circ f^{-1}$. The reasoning for this mapping is below

$$\begin{array}{ccc} [x, f(x)] & \xrightarrow{h} & [\tilde{x}, \tilde{f}(\tilde{x})] \\ f^{-1}\uparrow & & \downarrow\tilde{f} \\ [f(x), f^2(x)] & \xrightarrow{\overline{h}} & [\tilde{f}(\tilde{x}), \tilde{f}^2(\tilde{x})] \end{array}$$

Note that we have marked $\overline{h}$ to distinguish it from h, but these functions are actually the same except for their domain. We may repeat this process for extending the mapping to the sections $[f^n(x), f^{n+1}(x)] \to [\tilde{f}^n(\tilde{x}), \tilde{f}^{n+1}(\tilde{x})]$. So we can extend h in the region $[x, S)$ and also use backward iterations to extend h to the region $(N, x]$. With this extension, we have $h(S) = \tilde{S}$ and $h(N) = \tilde{N}$. We can perform a similar procedure to get a map on the other side of the circle. Q: Can we elaborate on what the similar procedure is? Also, once we have the extensions of h on both 'sides' of the circle, why can we combine them into one continuous map? Thus, through these extensions, we are able to get a map h which is a homeomorphism on the entire circle. This shows conjugacy since

$$h(f(y)) = \tilde{f} \circ h \circ f^{-1}(f(y)) = \tilde{f}(h(y))$$

■

Remark. In the general setting, there are two situation, we can have non-hyperbolic points (i.e. in rotations) and in this case, diffeomorphisms are not structurally stable. In the remaining case, we can have hyperbolic points. With some additional assumptions, the map will be structurally stable.

Remark. In terms of points, we can classify them into hyperbolic points and parabolic points in a general dynamical setting. In Hamiltonian mechanics, there is a further delineation between strictly hyperbolic and elliptic points.

Definition. *Topological Conjugacy for Flows*
Suppose we have two flows $\{f^t\}$ and $\{g^t\}$. We call these flows topologically conjugated if there exists a homeomorphism h which satisfies

$$f^t = h \circ g^t \circ h^{-1}$$

for all $t \in \mathbb{R}$

Remark. This above condition occurs only very rarely. This definition was given when dynamical systems was beginning as a discipline, which may explain why it is uncommon.

Definition. *Phase Portrait*
Every point x in the phase space belongs to some trajectory. These trajectories partition the phase space X, meaning we may write

$$X = \bigsqcup_{\text{orbits}} \{g^t(x)\}_{t\in\mathbb{R}}$$

Since the function g is defined on X, every x must belong to some orbit. This splitting is called a *phase portrait* (i.e. the set of all orbits).

Q: How do we show that if the intersection of two orbits is nonempty, then the orbits are the same?

Definition. *Orbital Topological Equivalence*
Suppose now that we have two phase portraits of two flows $\{f^t\}$ and $\{g^t\}$. Let $h : X \to X$ be a homeomorphism which satisfies

$$h(\{g^t(x)\}_{t\in\mathbb{R}}) = \{f^t(h^{-1}(x))\}_{t\in\mathbb{R}}$$

Then, these flows are *orbital topologically conjugated* and that h is the *orbital topological equivalence.* The meaning of the equation above is that all trajectories of one flow in g are sent to trajectories in some other flow h.

Definition. *Structural Stability of a Flow*
Let $\{g^t\}$ be a flow. We will consider structural stability with respect to orbital topological equivalence. Suppose there is a metric on the flow. The flow is *structurally stable* if for any flow that is close enough (i.e. in a neighborhood) is orbital topologically conjugated to $\{g^t\}$.

Remark. There is another way to characterize this. Suppose we have a manifold M and a vector field $\nu(M)$. This vector field yields a flow through autonomous differential equations when we consider solutions of $\dot{x} = \nu(x)$. In this case, we may think about structural stability of vector fields. We may define a neighborhood of the vector field $U(\nu)$ (with the metric being the maximum of the difference between two vectors at any point). In this neighborhood, we require that all flows generated by corresponding vector fields are orbital topologically equivalent. A vector field may be a more complicated object given that it is defined on the tangent bundle of M, but this is a common alternative way of thinking of structural stability.

Theorem. *Andronov-Pontryagin Theorem*
A vector field $\nu(\mathbb{S}^2)$ on the sphere is structurally stable if and only if all singular points (which correspond to fixed points of the flow) are hyperbolic, all cycles are hyperbolic, and there are no saddle connections.

Remark. These conditions are actually open and dense, so typical flows on a sphere are structurally stable.

4.5 Exercises

1. Let $\varphi \mapsto \varphi + \alpha$ and $\varphi \mapsto \varphi + \beta$ be two rotations. For which pairs (α, β) are these rotations conjugated?

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5.1 Circle Doubling Structural Stability

Structural stability is a property that is related to topological conjugacy.

Definition. *Structurally stable*
The map f is structurally stable in the set (or class) of maps $\mathscr{C}$ if there exists a neighborhood $U(f) \subseteq \mathscr{C}$ such that $g \sim f$ for all $g \in U(f)$. The equivalence relation $\sim$ is that of topological conjugacy.

It is not important which set $\mathscr{C}$ we choose, though it is often something like smooth maps or diffeomorphisms.
In our previous lecture, we proved that the North-South map is structurally stable. We will go further and prove that circle doubling is structurally stable. To begin, consider the map $\varphi \mapsto k\varphi$ for $k \in \mathbb{N}$ where $k > 1$. Our motivation for studying this is because it is insightful to first understand the notion of structural stability for maps on the circle, the second reason is that these one-dimensional dynamics represent a model for more complex pictures. Since this map is no longer a diffeomorphism, we will be considering $\mathscr{C} = \mathscr{C}^1(S^1)$, the set of $\mathscr{C}^1$ maps from the circle to itself. As a reminder, the metric in this space is

$$\|f - g\| = \sup_x |f(x) - g(x)| + \sup_x |f'(x) - g'(x)|$$

Proposition. The map E_k, given by $\varphi \mapsto k\varphi$ for $k \in \mathbb{N}$ where $k > 1$, is structurally stable in $\mathscr{C}^1(S^1)$.

Proof. We abbreviate $\mathscr{C} := \mathscr{C}(S^1)$. We will need to rely on a sequence of lemmas to arrive at our desired result.
Lemma. There exists a $U(E_k) \subseteq \mathscr{C}$ such that for all $f \in U(E_k)$, $f' > 1$ (i.e. f is a dilation), where f is orientation-preserving and has degree k.
First, we consider the projection map $\pi : \mathbb{R} \to S^1$ given by $x \mapsto \{x\}$. We have a lifting $F : \mathbb{R} \to \mathbb{R}$ for $f \in \mathscr{C}$ defined by the property $\pi \circ F = f \circ \pi$. It is always possible to obtain F from f, we can just consider repeating the function f with the same vertical shift every time the function repeats. We will have $F(1) - F(0) = k$ for $k \in \mathbb{Z}$ in this case, and we call k the degree of f. We must have $f(0) = f(1)$ since these points are identified in S^1, so this construction of F by vertical shifts is possible. Clearly, since $E_k' = k \geqslant 2$, it is clear that $f' > 1$ if we choose an appropriate neighborhood around E_k'. If all derivatives are positive, then the degree of the map will be a continuous function. Now, since degree is a continuous function and its possible values are integers, it must be locally constant. Hence f has degree

k. This concludes the lemma.

Now, given $f \in U(E_k)$, we want to show that $f \sim E_k$, that is, f is topologically conjugated to E_k. We will accomplish this in three steps

1. First we will build a map $h : S^1 \to S^1$ which satisfies

$$E_k \circ h = h \circ f$$

   and h will be continuous, surjective, and $\deg(h) = 1$.

2. We will obtain another map $\tilde{h} : S^1 \to S^1$ which satisfies

$$\tilde{h} \circ E_k = f \circ \tilde{h}$$

   Again, $\tilde{h}$ will be continuous, surjective, and $\deg(\tilde{h}) = 1$.

3. Finally, we will show $h \circ \tilde{h} = \text{id}$, which will imply that $E_k \sim f$.

We proceed by proving another lemma.

Lemma. f has a fixed point, $f(O) = O$.

We may construct a lifting $F : \mathbb{R} \to \mathbb{R}$ of f. We want to determine if there exists x such that $(F(x) - x) \in \mathbb{Z}$. Once we have this point, it is clear that the projection of this point will be a fixed point for f. We know that $F(1) - F(0) = k \geqslant 2$. Define a function

$$\tilde{F}(x) = F(x) - x$$

In this case, $\tilde{F}(1) - \tilde{F}(0) \geqslant 1$. So, the image $\tilde{F}([0, 1))$ will have a length that is greater than or equal to 1. As a result, there exists $x \in [0, 1)$ such that $\tilde{F}(x) = n$. Hence, we have found x satisfying $F(x) - x \in \mathbb{Z}$ and from this, we get the existence of the fixed point $f(O) = O$. This concludes the lemma. We will assume that our fixed point is 0, because we can conjugate by the rotation R_α given by $\varphi \mapsto \varphi + \alpha$ where $\alpha = -O$.

Now, we have $F(0) = 0$ and $F(1) = k$ since f is of degree k. Since F is continuous and strictly monotonic, we can find points $F(a_i) = i$ where $0 < a_1 < a_2 < \ldots < a_{k-1} < 1$. We may map any of these intervals into S^1

$$f([a_m, a_{m+1})) = [0, 1) = S^1$$

We intend to build a function H with the property $H(x + 1) = 1 + H(x)$ (i.e. it is surjective and of degree 1). We will want our construction to satisfy $H(0) = 0$ and $H(1) = 1$ eventually. Take a class of functions

$$Z = \{H \in \mathscr{C}^0([0, 1]), H(0) = 0, H(1) = 1\}$$

We define a special operator on this class $\mathcal{F} : Z \to Z$ by

$$\mathcal{F}(H)(x) = \begin{cases} \frac{1}{k} H(\{F(x)\}) + \frac{m}{k}, & x \in [a_m, a_{m+1}) \\ 1, & x = 1 \end{cases}$$

and we assume for simplicity that $a_k = 1$. We give a brief explanation for $\mathcal{F}$. In the first case, we have $F([a_m, a_{m+1})) = [m, m+1)$, applying H, we are left with $[0, 1)$. Multiplying this interval by $\frac{1}{k}$ and adding $\frac{m}{k}$ gives the interval

$$\left[\frac{m}{k}, \frac{m+1}{k}\right)$$

This shows that $\mathcal{F}$ is continuous, sends continuous functions to continuous functions, and sends 0 and 1 to themselves.

Moving on, suppose we have a fixed point of this operator $\mathcal{F}$, that is some H such that $\mathcal{F}(H) = H$. We claim that H will be the conjugacy we want. Observe that

$$\begin{aligned} E_k \circ h &= E_k \circ \pi \circ \mathcal{F}(H) = \pi \circ k\mathcal{F}(H) \\ &= \pi H(\{F(\cdot)\} + m) \\ &= h \circ \{F(\cdot)\} = h \circ f \end{aligned}$$

If $\mathcal{F}$ is a contraction, then there is a unique fixed point, which we may then take to be H to use in the above argument. To prove that $\mathcal{F}$ is a contraction, given the distance function ρ, we want to show

$$\begin{aligned} \rho(\mathcal{F}(H_1), \mathcal{F}(H_2)) &\leqslant \frac{1}{k} \max_{[0,1]}(H_1(\{F(x)\}) - H_2(\{F(x)\}) \\ &= \frac{1}{k} \cdot \rho(H_1, H_2) \end{aligned}$$

Since $k \geqslant 2$, we see that $\frac{1}{k} < 1$, so $\mathcal{F}$ indeed has a fixed point. We are done with the first step.

We define a new operator

$$\tilde{\mathcal{F}}(H) = \begin{cases} f_m^{-1}(H(\{kx\})), & x \in \left[\frac{m}{k}, \frac{m+1}{k}\right) \\ 1, & x = 1 \end{cases}$$

The structure of these operators is the same, notice that

$$E_k^{-1} H(F), \quad f_k^{-1} H(E_k)$$

in the first case and second case, respectively. Recall the map $f : [a_m, a_{m+1}) \to [0, 1)$. This map is a bijection, so we call its inverse map f_m^{-1}. We follow a similar sequence to that of first step to conclude the existence of a fixed point (via Mean Value Theorem), $\hat{H}$. The natural projection of $\hat{H}$ will be $\hat{h}$. We may show that $\hat{h} \circ E_k = f \circ \hat{h}$. This concludes the second step.

For the third step, we want to demonstrate

$$h \circ \hat{h} = \mathrm{id}$$

that is, $\hat{h} = h^{-1}$, which will give $E_k = h \circ f \circ h^{-1}$. Observe that

$$\underbrace{h \circ \hat{h}}_{g} \circ E_k = h \circ f \circ \hat{h} = E_k \circ \underbrace{h \circ \hat{h}}_{g}$$

Hence, $g \circ E_k = E_k \circ g$. It may be shown that $\deg(g) = 1$. We would like to show that g is the identity. Start with the lifting G. Consider $\tilde{G}(x) = G(x) - x$. Since g is of degree 1, we know that $G(1) - G(0) = 1$. Thus, $\tilde{G}(1) - \tilde{G}(0) = 0$. Hence, $\tilde{G}$ is a periodic function of the real line. Since $G(kx) = k(G(x))$, it follows that

$$\tilde{G}(kx) + kx = \tilde{G}(x) \cdot k + x \cdot k$$
$$\tilde{G}(kx) = k\tilde{G}(x)$$

By induction, we have $\tilde{G}(k^n x) = k^n \tilde{G}(x)$ for all $n \in \mathbb{N}$. Since $\tilde{G}$ is a periodic function on S^1, it must be bounded. If $\tilde{G}(x) \neq 0$, then we see that

$$\lim_{n\to\infty} \tilde{G}(k^n(x)) = \lim_{n\to\infty} k^n \tilde{G}(x) = \infty$$

which cannot happen since $\tilde{G}$ is bounded. Therefore, we must have $\tilde{G}(x) \equiv 0$ and $G(x) = x$ and finally $g = \text{id}$. ■

We give some brief remarks to recapitulate what happened in the proof.

Remark. Regarding the third step, it is a general phenomenon that if you have a map that is isotopic to the identity (i.e. it has degree 1 on S^1), and a commutation of this map with another map that is not isotopic to the identity and with nontrivial relations on the fundamental group of your manifold, then it is probable that this other map is the identity.

Remark. Regarding the first and second steps in showing topological conjugacy, these steps are similar and based on the contraction principle. The main difficulty in these steps was that our map was not a diffeomorphism (we eventually work with a diffeomorphism, the Anosov diffeomorphism). To solve this problem, we split our circle into intervals defined by the points $a_1, a_2, \ldots$ and our map will be locally a diffeomorphism between two consecutive points $[a_m, a_{m+1})$ and $[0, 1)$. After splitting our circle, we try to conjugate our map. We accomplish this by considering $\mathcal{F}$, which we prove is continuous and a contraction. The importance of the fixed point is that we will have the form

$$\mathcal{F}(h) = B^{-1} \circ h \circ A \implies h = B^{-1} \circ h \circ A \implies Bh = hA$$

5.2 Exercises

1. Show that global contraction on the real line $\mathbb{R}$ given by $x \mapsto \alpha x$ for $\alpha < 1$ is structurally stable in diffeomorphisms of the real line.

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We discuss an example of a diffeomorphism that is not realizable as a time one flow. Such a map should have periodic points that are not fixed and isolated. We begin by considering two maps on S^1. The first map f_1 has two attracting points and two repelling points. The four points are located on the cardinal directions and the ones on the vertical axis (north and south) are the repelling points. The second map f_2 will be rotation by a half circle, that is $\varphi \mapsto \varphi + \frac{1}{2}$. Now, consider their composition $f = f_2 \circ f_1$. Observe that for any non-fixed point x, we have $\omega(x) = \{\alpha, \alpha + \frac{1}{2}\}$, the two attracting points of f_1. Hence $\{\alpha, \alpha + \frac{1}{2}\}$ are two isolated, periodic points which are not fixed.
The next part of the lecture discusses measure properties of dynamical systems.

6.1 Invariant Measures

Consider a dynamical system $(X, f, \mathbb{Z})$, where X is the phase space, f our iteration function, and $\mathbb{Z}$ the time variable (discrete time). Further, let X be a topological space with measure μ. We will assume:

1. the measure μ is probabilistic, that is $\mu(X) = 1$;
2. μ is Borelian, meaning that all open sets are measurable.

Definition. *Invariant Measure*
An invariant measure under the measurable function $f : X \to X$ satisfies

$$\mu(A) = \mu(f^{-1}(A)) \text{ for all } A$$

Examples.

1. Consider the function $f(x) = \frac{x}{2}$ on the real line $\mathbb{R}$. Define the δ_0 measure by

$$\delta_0(A) = \begin{cases} 1 & \text{if } 0 \in A \\ 0 & \text{otherwise} \end{cases}$$

   This is an invariant measure because $0 \in f^{-1}(A) \iff 0 \in A$. In fact, this is the only possible invariant measure for this function. For instance, suppose there is another invariant measure μ under f, and take any interval I which does not include 0. If this interval has positive measure, we can iterate it through preimages to get a sequence of disjoint intervals (which all do not include 0), $I, f^{-1}(I), f^{-1}(f^{-1}(I)), \ldots$ But then

$$\sum_{k=0}^{\infty} \mu(f^{-k}(I)) = \infty$$

   contradicting that μ is a probability measure.

2. The rotation $\varphi \mapsto \varphi + \alpha$ on a circle. All rotations have at least one invariant measure, the Lebesgue measure from a geometrical point of view (if we consider the circle as a group, then we will have Haar measure).

   Do there exist other invariant measures? It depends on whether α is rational or not. Suppose that the rotation is rational: $\varphi \mapsto \varphi + \frac{n}{k}$. In this case, n does not matter. We split our circle into intervals $\frac{1}{k}$. On arbitrary intervals, we may define some measure, λ, and in any other interval, we have the same measure, except shifted. For example, we may define

   $$\lambda_j(A) = \lambda_0(R_{-\frac{j}{k}}(A))$$

   In contrast, for irrational rotations, we have the opposite situation. This is because there is a property called *unique ergodicity* which states that the map has only one invariant measure. It turns out that irrational rotation is uniquely ergodic, and this may be shown via the Kronecker-Weyl theorem. To be explicit, suppose that μ is an invariant measure for the rotation R_α, where α is irrational. By definition, we have

   $$\mu(I) = \mu(R_\alpha^{-n} I)$$

   Observe that

   $$\int_0^1 \chi_I \, d\mu = \int_0^1 \chi_{R_\alpha^{-n} I} \, d\mu = \int_0^1 \chi_I \circ R_\alpha^n \, d\mu$$

   Recall the function

   $$\mathcal{A}_n(\chi_I)(x) = \frac{\sum_{j=0}^n \chi_I(R_\alpha^j x)}{n+1}$$

   Hence

   $$\int_0^1 \mathcal{A}_n(\chi_I) \, d\mu = \frac{\sum \int_0^1 \chi_I \circ R_\alpha^n \, d\mu}{n+1} = \int_0^1 \chi_I \, d\mu = \mu(I)$$

   for any n. By uniform convergence, we see that

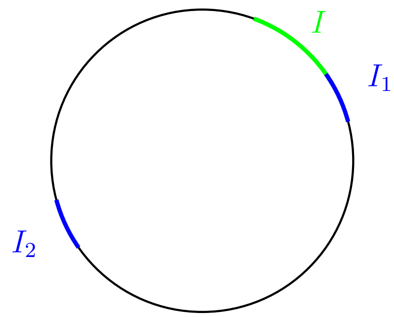
   $$\int_0^1 \mathcal{A}_n(\chi_I) \, d\mu \xrightarrow[n \to \infty]{} \int_0^1 |I| \, d\mu = |I|$$

   Thus, we see that our invariant measure coincides with length, meaning that it is the Lebesgue measure.

Definition. *Uniquely Ergodic*

A map f is uniquely ergodic if there exists only one invariant measure.

We now discuss the invariant measures of circle doubling (and related transformations). Let E_k be the map $\varphi \mapsto k\varphi$. To begin, it turns out that the Lebesgue measure is invariant under E_k. This may be counterintuitive if we think based on a similar argument we made in the first example above. However, recall that we are actually dealing with preimages. So, given any interval, there will be k preimages that are k times shorter in length.



E_2

Above, the green is an arbitrary interval while the blue are the preimages. Hence, if we add all the parts of the preimage, we will in fact get an interval of the same length, showing that the Lebesgue measure is, in fact, invariant. The map E_k is not uniquely ergodic, for instance, we also have

1. the δ_0 measure;
2. the sum of delta measures in any periodic orbit. Suppose E_k has an orbit of finite length, $x, E_k(x), \ldots, E_k^{n-1}(x)$ and $E_k^n(x) = x$. We may define

$$\mu = \frac{1}{n}\left(\delta_x + \delta_{E_k(x)} + \ldots + \delta_{E_k^{n-1}(x)}\right) = \frac{1}{n}\sum_{i=1}^{n} \delta_{ix}$$

   In fact, periodic orbits are dense for any covering.
3. Given two invariant measures μ_1, μ_2, we may take sums like $\frac{\mu_1+\mu_2}{2}$ as a new invariant measure.

However, we cannot simply shift the measure as with rational rotations.
Nonexistence of Invariant Measure
If the map is not continuous and only measurable, then it is possible for no invariant measures to exist. To illustrate this, consider the map $f : [0, 1] \to [0, 1]$

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Since $f^{-1}((\frac{1}{4}, \frac{1}{2})) = (\frac{1}{2}, 1)$, from the assumed invariance of μ, we have $\mu(\frac{1}{2}, 1) = \mu(\frac{1}{4}, \frac{1}{2})$. Similarly, we know that

$$\mu\left(\frac{1}{2}, 1\right) = \mu\left(\frac{1}{4}, \frac{1}{2}\right) = \ldots = \mu\left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right)$$

We must have

$$\mu\left(\bigcup_n \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right)\right) < \infty$$

But this is only possible if the measure of all of these intervals $(\frac{1}{2}, 1), (\frac{1}{4}, \frac{1}{2}), \ldots$ is 0. Moreover, we must have $\mu(1) = \mu\left(\frac{1}{2}\right) = \ldots = \mu\left(\frac{1}{2^n}\right) = 0$ (we are thinking of preimages of one point). These measures must all equal zero by the same finiteness argument we used for the measures of intervals. So, it follows that the measure of all half open intervals is zero and

$$\mu(0,1] = 0 \implies \delta_0(0) = 1 \implies \mu(1) = 1$$

This is a contradiction, so such a invariant, probabilistic measure μ cannot exist.

6.2 Krylov-Bogolyubov Theorem

Theorem. *Krylov–Bogolyubov Theorem*
Let X be a compact metric space and $f : X \to X$ be a continuous map. Then, there exists an invariant, probabilistic measure for f.

Remark. When we talk about topology, we usually start with calculus and then lead into purely dynamical results. In the case of measure, our results are based on functional calculus (i.e. theorems based on infinite dimensional spaces, and so on).

Theorem. *Riesz-Markov-Kakutani theorem*
Using the same setting as given in the Krylov-Bogolyubov theorem, let $\mathscr{C}(X)$ be the set of continuous functions on X. Let I be a bounded, positive linear functional on $\mathscr{C}(X)$. Then there exists a unique measure such that

$$I(\varphi) = \int_X \varphi \, d\mu \text{ for all } \varphi \in \mathscr{C}(X)$$

Essentially, any measure is a positive functional on continuous functions.
Proof (Krylov-Bogolyubov). We start by choosing any probabilistic measure. For instance, it could be δ_x, a random delta measure for some point x. We will denote the measure μ and define

$$\mu_n = \frac{1}{n+1}\left(\mu(\cdot) + \mu(f^{-1}(\cdot)) + \ldots + \mu(f^{-n}(\cdot))\right)$$

Notice that

$$|\mu_n(A) - \mu_n(f^{-1}(A))| = \frac{1}{n+1}\left|\mu(A) - \mu(f^{-n-1}(A))\right| \leqslant \frac{1}{n+1} \qquad (*)$$

So, our sequence of measures μ_n becomes more and more invariant as we increase n, which suggests that we should consider the 'limit'. To be rigorous, we start by defining

$$I_n(\varphi) = \int_X \varphi \, d\mu_n$$

Since our function is on a compact set with a metric, it is separable, so we can choose a set of functions $\{\varphi_1, \ldots, \varphi_k, \ldots\}$ which is dense in $\mathscr{C}(X)$. Since any continuous function on a compact set attains a maximum, we have

$$|I_n(\varphi_k)| \leqslant \max |\varphi_k|$$

This just follows from the integral definition and the fact that our measure is probabilistic (another way to see this is that the average of a function is less than the maximum). Showing the steps:

$$|I_n(\varphi_k)| = \left| \int_X \varphi_k \, d\mu_n \right| \leqslant \int_X |\varphi_k| \, d\mu_n \leqslant \int_X \max |\varphi_k| \, d\mu_n = \max |\varphi_k|$$

Moving on, we may enumerate our functions as follows

$$\begin{aligned} &I_1(\varphi_1), \ldots, I_1(\varphi_k), \ldots \\ &I_2(\varphi_1), \ldots, I_2(\varphi_k), \ldots \\ &\vdots \end{aligned}$$

So, we have a diagonal process. In each row, we may find a convergent subsequence, that is, there exists a subsequence $\{n_j\}$ such that for all k

$$\lim_{j \to \infty} I_{n_j}(\varphi_k) = J_k$$

Now, we set

$$I_\infty(\varphi_k) = J_k$$

Further, we may continue our functional using the Hahn-Banach theorem, that is, there is a bounded, positive linear functional I_∞ on $\mathscr{C}(X)$. Thus, we may deduce, by the Riesz-Markov-Kakutani theorem, that I_∞ is representable by some measure, call it μ_∞. We must show that this measure is invariant. Observe that

$$\mu_\infty(A) = \int_X \chi_A \, d\mu_\infty = I_\infty(\chi_A) = \lim_{n \to \infty} I_n(\chi_A) = \lim_{n \to \infty} I_n(\chi_A \circ f) = \mu_\infty(f^{-1}A)$$

where we have the equality $\lim_{n \to \infty} I_n(\chi_A) = \lim_{n \to \infty} I_n(\chi_A \circ f)$ from the equation labeled $(*)$ above. Therefore, our measure μ_∞ is invariant, as claimed.

■

Q: Did we need the density of φ_k to extend I_∞ via Hahn-Banach?

Q: How did we justify $\lim_{n \to \infty} I_n(\chi_A) = \lim_{n \to \infty} I_n(\chi_A \circ f)$?

A: To see this equality, notice that

$$|I_n(\chi_A) - I_n(\chi_A \circ f)| = \left| \int_X \chi_A \, d\mu_n - \int_X \chi_{f^{-1}(A)} \, d\mu_n \right| = |\mu_n(A) - \mu_n(f^{-1}(A))| \leqslant \frac{1}{n+1}$$

To summarize our discussion of the proofs above, the key point is the contrast between measurable maps that are continuous and those which are not continuous. If a measurable map is not continuous, then it may not have an invariant probabilistic measure. However, if a measurable map is continuous, then by Krylov-Bogolyubov, it has an invariant probabilistic measure. We discuss some more examples and applications now.

6.3 Extremal Points and Ergodicity

Proposition. Let f be the North-South map. We may describe the set of invariant measures by

$$\alpha\delta_N + (1-\alpha)\delta_S, \quad \alpha \in [0,1]$$

Proof. First, we note that δ_N and δ_S are invariant. Let A be some measurable set. To see this for δ_N, we have $N \in f^{-1}(A) \iff N \in A$. Hence $\delta_N(A) = \delta_N(f^{-1}(A))$ for all A. The same argument applies for δ_S. Now, we show that all invariant measures must have the form stated in the proposition. Let μ be an invariant measure and (S,a) some interval on the circle. Due to the invariance of the measure, we have $\mu(S,a) = \mu(S, f^{-1}(a))$. Since measure is an additive function, we have

$$\mu(S,a) + \mu([a, f^{-1}(a))) = \mu(S, f^{-1}(a)) \implies \mu(a, f^{-1}(a)) = 0$$

We may conclude, by induction, that $\mu(f^{-k}(a), f^{-k-1}(a)) = 0$ for all $k \in \mathbb{N}$. It may also be shown (via the inverse map, i.e. switch the north and south poles) that $\mu(f^k(a), f^{k-1}(a)) = 0$, so that this equality holds for all integers. Thus

$$\mu\left(\bigcup_{n\in\mathbb{Z}}(a, f^{-1}(a))\right) = 0$$

But this union covers almost all points of the North-South map; it only excludes the points in the orbit of a. Since a is arbitrary, we can apply the same argument for a distinct point b from a to reach the conclusions that $\mu((N,S)\setminus \operatorname{orb}(a)) = 0$ and $\mu((N,S)\setminus \operatorname{orb}(b)) = 0$. Taking the union of these sets, we see that $\mu(N,S) = 0$. Since the argument applies to the other side of the circle as well, we see that $\mu(S,N) = 0$. Therefore, for any invariant measure under f, there are only two points where there is a support of this measure: $\operatorname{supp}\mu = \{S, N\}$. This forces the set of measures to have the form in the proposition.

■

Remark. The purpose of understanding the proposition above is to illustrate a more general idea: if μ_1, μ_2 are invariant measures, then $\alpha\mu_1 + (1-\alpha)\mu_2$ is also an invariant measure (if there are more measures, then it is a convex hull). The set of all invariant measures is a convex set in the space of probabilistic measures.

Remark. There is an important idea in complex analysis called **extremal points**.

Definition. *Extremal Point*
Let x, y, z be points of a set S and suppose that $y \neq z$ and

$$x = \alpha y + (1-\alpha)z, \quad \alpha \in [0,1]$$

The point x is *extremal* if $\alpha = 0$ or $\alpha = 1$.

In general, infinite-dimensional convex sets do not necessarily have extremal points. However, for measures there is a special theorem.

Theorem. *Choquet's Theorem*
Let S be a convex, compact subset in a space of probabilistic measures on X. Then, there are extremal points S_{ext}. Moreover, for any measure $\mu \in S$, there exists ν on S_{ext} such that

$$\mu = \int_{S_{\text{ext}}} \Theta \, d\nu(\Theta)$$

We call points in the set S_{ext} **ergodic measures**.

As a clarifying point to the notation above, note the measure μ is a point in the space of Borelian measures while ν is a measure on a space of measures. There is a more elementary characterization of ergodic measures. The theorem above gives motivation on why ergodic measures are important, since all other measures are a (possibly infinite) sum of ergodic measures.

Definition. *Ergodic Measures*
We say that a measure μ is ergodic with respect to a function f if any invariant measurable set either has measure zero or measure one. In symbols, for all A satisfying $f^{-1}(A) = A$, we have $\mu(A) = 0$ or $\mu(A) = 1$.

Remark. From the Krylov-Bogolyubov theorem and Choquet's theorem, any continuous and measurable map should have ergodic, invariant measures.
Examples. There are some familiar objects with ergodic measures.

1. Consider irrational rotation on the circle. Recall that we have shown that irrational rotation is uniquely ergodic. It is easier to see ergodicity from Choquet's theorem since there is only one invariant measure so that it must be an extremal point.

2. Consider the North-South map. We have shown that all invariant measures under this map must have the form $\alpha\delta_N + (1-\alpha)\delta_S$ for $\alpha \in [0,1]$ in an earlier proposition. Hence, only δ_N and δ_S are ergodic.

3. Consider the doubling map on the circle. There are many ergodic measures. First, δ_0 is ergodic since any set may only have measure zero or one. Lebesgue measure is also ergodic relative to doubling (this is less obvious). There is a duality between Fourier series (sines and cosines) and Lebesgue measure.

Proposition. The Lebesgue measure is ergodic relative to circle doubling, E_2

Proof. Let A be an invariant set for $\varphi \mapsto 2\varphi$. We may represent χ_A, a measurable function, as a Fourier series almost everywhere:

$$\chi_A(x) = \sum_n c_n e^{inx}$$

Under the map, we have

$$\chi_{2A}(x) = \sum_n c_n e^{2inx}$$

But these functions must be equal since A is invariant. Assume that $c_k \neq 0$ for some $k \neq 0$. By matching coefficients for e^{2inx}, it follows that $c_{2k} = c_k$ and by iterating this argument $c_{2^n k} = c_k$. But then $\lim_{n\to\infty} c_n \neq 0$ so that the series does not converge, which is a contradiction. This shows our assumption was incorrect and we conclude that only c_0 is possibly nonzero. Hence, we must have $\chi_A(x) = 1$ or $\chi_A(x) = 0$, almost everywhere.

■

Proposition. All extremal points of a set of invariant measures are ergodic.

Proof. Let μ be a measure that is not ergodic. So, we have $0 < \mu(A) < 1$ for some invariant set A. Define

$$\mu_A(B) = \frac{\mu(B \cap A)}{\mu(A)} \text{ and } \mu_{\overline{A}}(B) = \frac{\mu(B \cap \overline{A})}{\mu(\overline{A})}$$

where $\overline{A}$ indicates the complement. These measures are invariant measures for f since A is an invariant set for f. We may now write

$$\mu = \alpha\mu_A + (1-\alpha)\mu_{\overline{A}}$$

where $\alpha = \mu(A)$. Since $0 < \mu(A) < 1$, this combination is nontrivial. So, our measure was not an extremal point. ■

We discuss one more example. Recall Σ_2^+, the set of one-sided sequences consisting of 0 and 1 and the shift operator σ. We may define a subset of the phase space called a **cylinder**

$$C = \{\Sigma_2 \mid x_1 = 0, x_{100} = 1, x_{105} = 0\}$$

The cylinder is a set where a finite number of indices in the sequence are fixed. In the case above, where there are three fixed sequences and there are two possible digits, we define the measure as $\mu(C) = \frac{1}{2^3} = \frac{1}{8}$. If there are n conditions (n fixed indices), then the measure is $\mu(C) = \frac{1}{2^n}$. In a more general setting, where there are k digits and n fixed indices, then we have $\mu(C) = \frac{1}{k^n}$ as our measure. This measure is called the **Bernoulli measure**. It may be shown that this measure μ is invariant with respect to σ and Σ_2.

6.4 Exercises

1. Describe $\sigma, \Sigma_2, \tilde{\mu}(C) > 0$ but which is not the Bernoulli measure.

7 March 5, 2025

7.1 Linear Anosov Diffeomorphisms

Historically, these systems were not invented by Dmitri V. Anosov, but Anosov made significant contributions towards an understanding of properties of these systems (structural stability). We start our discussion from something more basic: Fibonacci numbers.

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Recall that the generic formula for the n^{th} number of the Fibonacci sequence is

$$x_{n+2} = x_{n+1} + x_n$$

There is a special technique that allows us to solve these equations, which are called **recurrence relations**. In general, we have

$$a_m x_{n+m} + a_{m-1} x_{n+m-1} + \ldots + a_0 x_n = 0$$

The idea is that a term in the sequence is defined by the previous m numbers. If we assume that this equation has solutions of the form $x_n = \lambda^n$ and plug this into the recurrence, we have a polynomial condition

$$\sum_{i=0}^{m+1} a_i \lambda^i = 0$$

We want to find exactly $m + 1$ solutions, as this is a polynomial of degree $m + 1$. Denote these solutions $\lambda_1, \lambda_2, \ldots, \lambda_{m+1}$. Because the recurrence is linear, linear combinations of these solutions will also be solutions

$$x_n = \sum_{j=0}^{n+1} c_j \lambda_j^n$$

As an example, consider how this works for the recurrence relation of the Fibonacci numbers. Suppose λ is a solution. Then we have the equation $\lambda^2 = \lambda + 1$ or $\lambda^2 - \lambda - 1 = 0$ when we plug into the recurrence relation. The discriminant is $\sqrt{5}$ and we have

$$\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

There are standard names for these numbers. The positive number is denoted φ and is called the *golden ratio*, while the negative one is equal to $-\frac{1}{\varphi}$. Since we have these solutions, the general solution will have the form

$$x_n = c_1 \varphi^n + c_2 \left(-\frac{1}{\varphi}\right)^n$$

Given this form, we see that

$$x_0 = c_1 + c_2 = 0$$
$$x_1 = c_1\varphi - \frac{c_2}{\varphi} = 1$$

With the system of equations above, we may solve for c_1 and c_2. The situation differs when the polynomial has multiple (repeated) roots.

Remark. In general, there is a deep connection between polynomial equations and linear Anosov diffeomorphisms. We might expect this connection since the Anosov diffeomorphism describes something about the smooth structure on a manifold, but linearity imposes some algebraic structure. One specific case is the connection to continued fractions.

7.2 Example and Discussion of Anosov Diffeomorphisms

We can represent Fibonacci numbers in a different way:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} x_{n+2} \\ x_{n+1} \end{pmatrix}$$

It can be shown that the eigenvalues of this matrix are precisely φ and $-\frac{1}{\varphi}$. However, this matrix has one issue: its determinant is negative. The easiest way to deal with this is to take the square:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = A$$

This gives us the simplest example of a linear Anosov diffeomorphism. This system is commonly called **Arnold's cat**. To start understanding more about this transformation, we first note that this matrix acts on the plane $\mathbb{R}^2$.

1. An important property is that all of its entries are integers.

2. Second, the determinant of this matrix is 1. It is a square-preserving map. In particular, this map sends $\mathbb{Z}^2$ to $\mathbb{Z}^2$.

3. Further, if we look at the inverse map, we will see that all entries are also integers. Therefore, for any integer point, there is an integer preimage.

If one has a map on a plane which preserves all integer points, such a map may be factorized. Hence, there exists a factor map for A on this manifold, $\mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2$ mapping into $\mathbb{T}^2$. As an example of how this mapping acts on the torus

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{2}{5} \\ \frac{2}{5} \end{pmatrix} = \begin{pmatrix} \frac{6}{5} \\ \frac{4}{5} \end{pmatrix} = \begin{pmatrix} \frac{1}{5} \\ \frac{4}{5} \end{pmatrix}$$

where the last equality comes from how points are identified on $\mathbb{T}^2$. In theory, we can do the same for any linear map, but if our factorization is incorrect, we do not obtain a bijection. In this case it is a linear diffeomorphism. In principle, we may also have $\mathbb{R}^n / \mathbb{Z}^n : \mathbb{T}^n \circlearrowleft A$. In dimension 2, there are many maps of this type, our only conditions are that the map should consist of integer entries and have determinant 1. More generally, one should be careful to ensure that $|\lambda_i| \neq 1$ for any eigenvalue λ_i and that A, A^{-1} have integer entries.
We turn our discussion now to the number of periodic points. In our example, there are infinitely many periodic points, and the set of periodic points is dense. If we look at any point with rational coordinates under our map

$$A \underbrace{\left(\frac{p_1}{q}, \frac{p_2}{q}\right)}_{Q} = \left(\frac{2p_1 + p_2}{q}, \frac{p_1 + p_2}{q}\right)$$

Consider the orbit $\{Q, AQ, A^2Q, \ldots\}$. This orbit must be finite because we do not simplify denominators (we leave it as q). Hence, there are a maximum of q^2 possible points in our orbit. At some point in this orbit, we will have only periodic points. For example, taking $Q = \left(\frac{2}{5}, \frac{2}{5}\right)$ and A as $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, we have the sequence of points

$$\begin{pmatrix} \frac{2}{5} \\ \frac{2}{5} \end{pmatrix}, \begin{pmatrix} \frac{1}{5} \\ \frac{4}{5} \end{pmatrix}, \begin{pmatrix} \frac{1}{5} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{2}{5} \\ \frac{1}{5} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{3}{5} \end{pmatrix}, \begin{pmatrix} \frac{3}{5} \\ \frac{3}{5} \end{pmatrix}, \begin{pmatrix} \frac{4}{5} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{2}{5} \end{pmatrix}, \begin{pmatrix} \frac{2}{5} \\ \frac{2}{5} \end{pmatrix}$$

The pattern will repeat. So in this case, all points are periodic. For any points of rational coordinates, they will be periodic. Further, the rationals are dense, and therefore, we have a dense set of periodic points. This matrix has irrational eigenvalues and eigenvectors. In theory, a matrix of this type cannot have rational eigenvalues.
In our case, the eigenvectors will differ from each other by $\frac{\pi}{2}$. The orthogonality of these eigenvectors is because our matrix is symmetric.
Suppose we have a torus as our manifold, $M = \mathbb{T}^2$. At any point of this manifold, we have a tangent bundle, which we represent as the sum of two distributions.

$$TM = E_S M \oplus E_U M$$

where subscripts S and U refer to stable and unstable. We will have a split that occurs along two lines through the point on M. This same split occurs in an tangent space. In one direction the vectors of the tangent space are contracting and in another direction the vectors are expanding, that is

$$\|A^n v\| < C\varphi^{2n} \|v\|$$
$$\left\|A^{-n} w\right\| < C\varphi^{2n} \|w\|$$

where $v \in E_S$ and $w \in E_U$. So, we have infinitely many periodic points and all these periodic points are saddles, since we have eigenvalues where one is greater than 1 and the other less than 1.

Aside. We prove some facts.

1. The eigenvectors of a real symmetric matrix are orthogonal.
   Let A be a real symmetric matrix. If λ_1, λ_2 are distinct eigenvalues of A and x, y the corresponding eigenvectors, we have

$$\lambda_2 x^T y = x^T A y = y^T A^T x = y^T A x = \lambda_1 y^T x$$

   Hence $(\lambda_1 - \lambda_2) x^T y = 0$ by the commutativity of the dot product, and so $x^T y = 0$, showing x and y are orthogonal.

2. $\|A^{-n} w\| < C\varphi^{2n} \|w\|$ means w is expanding under A. Observe that

$$\|w\| = \left\|A^n \left(A^{-n} w\right)\right\| \leqslant \|A^n\| \left\|A^{-n} w\right\| < \|A^n\| C\varphi^{2n} \|w\|.$$

We summarize our discussion with a definition.

Definition. *Linear Anosov Diffeomorphism*
A *linear Anosov diffeomorphism* is a specific type of dynamical system defined on a compact manifold (often a torus) that exhibits uniform hyperbolic behavior. More precisely, if we consider the n-torus

$$\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n,$$

a linear Anosov diffeomorphism is an automorphism induced by a linear map

$$A : \mathbb{R}^n \to \mathbb{R}^n,$$

with

$$A \in \mathrm{GL}(n, \mathbb{Z}),$$

satisfying the following condition:

- **Hyperbolicity:** None of the eigenvalues of A lie on the unit circle in the complex plane, i.e., for every eigenvalue λ of A, $|\lambda| \neq 1$.

This condition guarantees that the tangent bundle of the torus splits into two invariant subbundles:

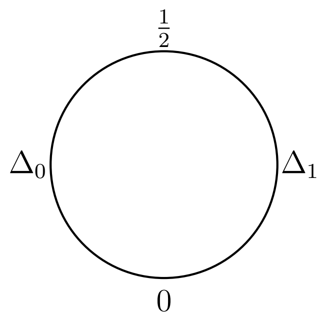
$$T\mathbb{T}^n = E^s \oplus E^u,$$

where:

- E^s is the **stable subspace**, in which vectors are uniformly contracted by the map.
- E^u is the **unstable subspace**, in which vectors are uniformly expanded by the map.

7.3 Measure Properties of Anosov Diffeomorphisms

Previously, we discussed how to calculate periodic points for circle doubling or tripling, E_2, E_3. We said that it is easier to make this calculation by changing the base of our number system. It is possible to think about this in more geometrical terms.
Suppose we have a circle, with 0 marked at the south and $\frac{1}{2}$ marked at the north. We split the circle into two sets which lie on the left and right of these points, $\Delta_0 = [0, \frac{1}{2}]$ and $\Delta_1 = [\frac{1}{2}, 0]$.



We can now describe a map called the Fate map which maps from M to Σ_A given by $x \mapsto (a_n)$, where x is a point on the circle and (a_n) is a sequence defined by

$$f^n(x) = \begin{cases} 0, & \text{if } f^n(x) \in \Delta_0 \\ 1, & \text{if } f^n(x) \in \Delta_1 \end{cases}$$

Q: But what happens if the point is in both sets? Note that in our case, the Fate map will go from M to Σ_2^+. In the case of circle doubling, the map will correspond to a Bernoulli shift since

$$x \mapsto a_0 a_1 a_2 \ldots$$
$$f(x) \mapsto a_1 a_2 a_3 \ldots$$

So this is another way to describe our correspondence between points and numbers. This correspondence and related research on spaces of sequences is called *symbolic coding*, and it is a useful tool in dynamical systems. Often in symbolic dynamics, the maps will be clear while the phase space will be harder to determine.
Continuing, we define another object connected to Anosov diffeomorphisms. We start with a matrix called the *stochastic matrix*, it is a square matrix where many coefficients are zero

$$\Pi = \begin{pmatrix} \pi_{11} & 0 & \pi_{13} & \ldots \\ \pi_{21} & 0 & 0 & \ldots \\ \pi_{31} & 0 & 0 & \ldots \\ \vdots & \ddots & \ldots & \end{pmatrix}^T$$

Also, we require $\pi_{ij} \geqslant 0$ and that $\sum_i \pi_{ij} = 1$. For concreteness, let us say that Π is an $N \times N$ matrix.

$$\Omega_\Pi = \{\omega \in \Sigma_N \mid \pi_{\omega_n \omega_{n+1}} > 0, n \in \mathbb{Z}\}$$

Recall that Σ_N is an infinite sequence in both directions, where values of the sequence are in $\{0, 1, \dots, N-1\}$. What is the point of this matrix? The idea is we have some digit, say 2. Within our stochastic matrix, the π_{2j} actually represent probabilities that the next element in the sequence will be j. So, the Ω_Π is the set of sequences that are allowed based on the probabilities in the stochastic matrix Π. The left shift on this sequence is denoted σ_Π. This system is called the *Markov chain*.

So, what was the point of this description? Well, these numbers π_{ij} define an invariant measure on this space. To explain this, we need a major theorem, the Perron-Frobenius theorem.

Theorem. *Perron-Frobenius Theorem*

Every stochastic matrix Π has an invariant vector, that is, a vector p such that $\Pi p = p$ where all coordinates of p are nonnegative. Also, this matrix cannot have eigenvalues which are greater than 1.

If all elements of Π^m are positive, then this matrix Π is called *transitive*. In the case where Π is transitive, then there exists a unique p with positive entries and is invariant, and all other eigenvalues are less than 1.

Aside. Invariance of Measures. Recall that a measure μ is invariant under a transformation T if

$$\mu(T^{-1}(A)) = \mu(A)$$

for all measurable sets A. For Markov chains, as we will elaborate on in the subsequent discussion, the transformation corresponds to the left shift operator which advances the sequence of states by one step.

We move to Markov invariant measures, which are defined on a specific type of cylinder. A cylinder is given by

$$C_i^m = \{* * \underbrace{\alpha_i \alpha_{i+1} \dots a_{i+m}}_{\Omega_\Pi} * *\}$$

Recall Π is an $N \times N$ matrix and p is an $N \times 1$ vector, with components $p_0, p_1, \dots, p_{N-1}$.

Definition. *Markov Invariant Measure*
Let μ be a measure. We define

$$\mu(C_i^m) = p_{\alpha_i}\pi_{\alpha_i\alpha_{i+1}}\cdots\pi_{\alpha_{i+m-1}\alpha_{i+m}}$$

where p_{α_i} is the α_i-component of our invariant vector p.

We should check the invariance of this measure under the left shift transformation (which should follow by definition) and if the measure is consistent. For an arbitrary β, we should have

$$\sum_\beta \mu(C_{\beta\alpha_0\ldots\alpha_m}) = \mu(C_{\alpha_0\ldots\alpha_m})$$

To explain this requirement with an example, consider a case where we have only two numbers, then

$$\mu(0011001) = \mu(10011001) + \mu(00011001)$$

This is because any 8-digit sequence with 0011001 as its final digits must begin with either 0 or 1. The set of all 8-digit sequences is just a disjoint union of these two cases.
Now, we want to show

$$\sum_\beta \mu(C_{\beta\alpha_0\ldots\alpha_m}) = \mu(C_{\alpha_0\ldots\alpha_m})$$

Making substitutions using the definition of the measure, we have

$$\sum_\beta p_\beta\pi_{\beta\alpha_0}\cdots = p_{\alpha_0}\pi_{\alpha_0\alpha_1}\cdots$$

Owing to the common terms on both sides of the equation, it is sufficient to show

$$\sum_\beta p_\beta\pi_{\beta\alpha_0} = p_{a_0}$$

To show this, observe that

$$\sum_\beta p_\beta\pi_{\beta\alpha_0} = \alpha_0(\Pi p) = \alpha_0(p) = p_{\alpha_0}$$

where the function α_0 stands for the α_0 projection and where the second equality follows from the invariance of p. Note also that the matrix entries have reversed row and columns from their usual sense. Hence, the measure is consistent.

7.4 Exercises

1. Let $\lambda_1, \lambda_2, \ldots$ be distinct roots. There may be more roots, but we will focus on the first two roots. Suppose that these roots have the form $\lambda_1, \lambda_1 + \varepsilon, \ldots$, and we try to consider what happens when $\varepsilon \to 0$. Show why assuming $x_n = c_j\lambda_j^n$ no longer works for the first two roots, and suggest how to remedy this.

2. Is it possible to represent geometric and arithmetic progressions in the form of a recurrent formula. Obtain the formula of the n^{th} term of the series.

8 March 5, 2025

We spend some time at the beginning of the lecture discussing the following pictures from Katok and Hasselblatt (p.84-85):

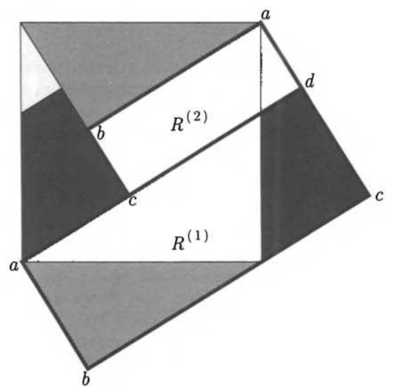


FIGURE 2.5.4. Partitioning the torus

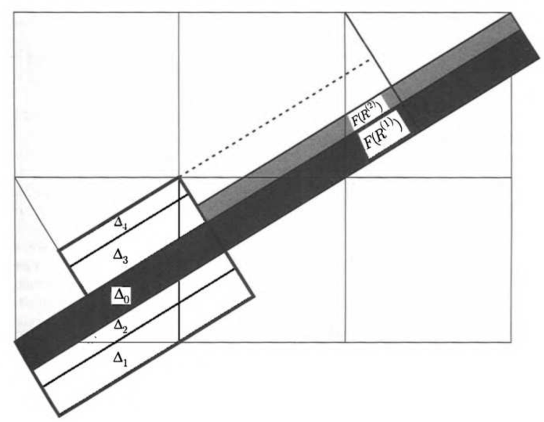


FIGURE 2.5.5. The image of the partition

Suppose we have a space

$$\Omega_{\Pi_A} = \{\omega \in \Sigma_5 : \pi_{\omega_n \omega_{n+1}} > 0\}$$

where $\Pi_A = \{\pi_{ij}\}$. We are using the subscript A to suggest a connection with the Anosov diffeomorphism. We can make this connection explicit by setting

$$\Pi_A = \{\pi_{ij}\} = \frac{\mu(A(\Delta i) \cap \Delta j)}{\mu(\Delta i)}$$

where Δi and Δj indicate rectangles corresponding to the image above and where μ is the Lebesgue measure. We note that it is possible to have chosen another invariant measure, as long as we are not on the boundaries of our rectangles. For instance, we could choose a random periodic orbit and some equidistributed measure on this periodic orbit. Continuing, the matrix will roughly appear as

$$\Pi = \begin{pmatrix} * & * & 0 & * & 0 \\ * & * & 0 & * & 0 \\ * & * & 0 & * & 0 \\ 0 & 0 & * & 0 & * \\ 0 & 0 & * & 0 & * \end{pmatrix}$$

where the symbol $*$ indicates a positive value. There is an eigenvector p for this matrix. This matrix may be iterated forward by some amount so that it will be transitive, for instance, Π^{100} may be transitive (all entries are positive). Hence, it will have a unique eigenvector and unique invariant Markov measure. The key idea is that any stochastic matrix of this type gives us an invariant measure, and this measure on the space of shifts of Markov chains corresponds to a measure on Anosov diffeomorphisms. This gives us a way to understand the behavior of Anosov diffeomorphisms.

To reiterate, we defined the entries of the stochastic matrix with a measure μ, Lebesgue or otherwise:

$$\{\pi_{ij}\} = \frac{\mu(A(\Delta i) \cap \Delta j)}{\mu(\Delta i)}$$

We are able to get a transitive matrix Π^m from the stochastic matrix with these entries (which depend on our measure). From the Perron-Frobenius theorem, we can get an invariant eigenvector p through which we can then define the Markov invariant measure. With rare exceptions, there is a bijection between measures on Anosov diffeomorphisms and Markov chains.

8.1 Proving the Perron-Frobenius theorem

We restate the theorem.

Theorem. *Perron-Frobenius Theorem.*
For every stochastic matrix, there exists a vector p satisfying $\Pi p = p$ and $p \geqslant 0$; for all other eigenvalues of the matrix, we have $|\lambda_i| \leqslant 1$. If $\Pi^m > 0$ (i.e. is a transitive matrix), then there exists a unique vector p such that $\Pi p = p$ and $p > 0$; for all other eigenvalues of the matrix, we have $|\lambda_i| < 1$.

Remark. Although we deal with stochastic matrices, where $\sum_i \pi_{ij} = 1$, it is possible to generalize this theorem to other matrices with positive coefficients by changing the property of invariance to maximal eigenvalue.

Remark. There is an alternate proof that depends on a choice of norms that forces a contraction mapping, but we follow another argument.

Proof of the Perron-Frobenius Theorem.
In this proof, our convention will be that the matrix's column entries sum to 1. Consider the set $P = \{x_i \geqslant 0 : i \in \{1, \dots N\}\}$, where p is a vector of length N. To clarify, if we have a two-dimensional vector, then P will be the first quadrant. Now define the space

$$\Sigma = \left\{(x_1, \dots, x_N) : x_i \geqslant 0, \sum x_i = 1\right\}$$

Notice that any vector from Σ is invariant under the stochastic matrix. To see this, observe that for some transformed vector $(y_1, y_2, \dots, y_N)$, we have

$$\sum_{i=1}^{N} y_i = \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_{ij} x_j = \sum_{j=1}^{N} x_j \sum_{i=1}^{N} \pi_{ij} = \sum_{j=1}^{N} x_j = 1$$

Hence, $\Pi\Sigma \subseteq \Sigma$. Further, note that Σ is the the standard simplex (i.e. in $\mathbb{R}^3$, it is the tetrahedron). Since it is a simplex, we may define this as the convex hull of some points, $\operatorname{conv}\{s_1, \ldots, s_N\}$. Since our map is linear, we may define the image $\Pi\Sigma$ as a convex hull of the image points $\{\Pi s_1, \ldots, \Pi s_N\}$. Thus

$$\Pi^n\Sigma = \operatorname{conv}\{\Pi^n s_1, \ldots, \Pi^n s_N\}$$

As we know, the intersection of convex sets is still convex. Define

$$\sigma = \bigcap_{n=0}^{\infty} \Pi^n\Sigma$$

It follows that σ is a convex, closed, nonempty, and invariant set. We claim that σ is again a simplex (potentially of smaller dimension). So, σ has less than or equal to N extremal points. To be explicit, we may consider the sequences of points $\{\Pi^n s_j\}$ (there are N such sequences). Since these sequences exist within a closed and bounded set, there exists a subsequence such that

$$\lim_{k\to\infty} \Pi^{n_k} s_j = \tilde{s_j}$$

Every $x \in \sigma$ may be written as a linear combination of $\tilde{s_j}$. Now, we will relabel the extremal points of σ as $p_1, \ldots, p_k$. Since these are extremal points, $\Pi : \{p_1, \ldots, p_k\} \to \{p_1, \ldots, p_k\}$ must be a permutation of these points. Consider the center of mass vector

$$p = \frac{p_1 + \ldots + p_k}{k}$$

It follows that $\Pi p = p$, showing that we have found our desired invariant vector.

Continuing, let $\kappa > 1$ be a real eigenvalue of Π and f be the associated eigenvector. We have

$$\frac{\Pi^n(p + \varepsilon f)}{\kappa^n} = \frac{p + \varepsilon\kappa^n f}{\kappa^n} \xrightarrow[n\to\infty]{} \varepsilon f$$

We observe that ε may be chosen so that $p + \varepsilon f$ belongs to P (i.e. even if f has some negative components, since $p \in P$ and ε is arbitrary, we can choose ε small enough so that the sum $p + \varepsilon f \in P$). Now, since Π is a stochastic matrix, its entries are all nonnegative, hence, $\Pi^n(p + \varepsilon f) \in P$ for all $n \in \mathbb{N}$ (i.e. P is invariant under Π). Since P is closed, this shows that $\varepsilon f \in P$ as well. Hence, $f \in P$. Finally, we must have $f \notin \Sigma$. If $f \in \Sigma$, then $\Pi^n f = \kappa^n f \in \Sigma$. But $\kappa^n f$ is not bounded while Σ is bounded. This shows that Π does not have real eigenvalues greater than 1.

On the other hand, suppose that $\kappa \in \mathbb{C}$ and $|\kappa| > 1$. We may write $\kappa = \rho e^{2\pi i\varphi}$. Let κ be an eigenvalue for Π and f its eigenvector. We may regard κ as a dilation and rotation in some plane. Let $p \in P$ and $\tilde{f}$ belong to this two-dimensional invariant space. Consider that

$$\frac{\Pi^n(p + \tilde{f})}{\rho^n}$$

will tend to this invariant plane (though it will be rotating as n increases). If φ is irrational, we have an irrational rotation of these points that get closer and closer to the plane. If φ is rational, the rotation is periodic and at some point we will be close to the original vector on the invariant plane. So, we have the same contradiction as in the case of the real eigenvalue, except instead of tending to a vector, we tend to a plane.

Aside. We elaborate on some details. Given the eigenvalue $\kappa \in \mathbb{C}$, we know that its complex conjugate will also be an eigenvalue, so denote this pair κ_1, κ_2. We are saying that in the space $\mathbb{R}^N$, there exists a plane which we may denote $\mathbb{R}^\Sigma$ which is invariant under the application of our matrix Π. We take $\tilde{f} \in \mathbb{R}^\Sigma$.

1. If z is a complex eigenvalue for some linear transformation T, then so is $\overline{z}$. Notice that
$$Tv = zv \implies \overline{T}\overline{v} = \overline{zv} \implies T\overline{v} = \overline{zv}$$
since T has real entries in our case.

2. The conjugate eigenvalues induce an invariant plane. For the eigenvector $x + iy$, we have $\Pi(x+iy) = \Pi(x) + i\Pi(y) = \rho e^{2\pi i \varphi}(x+iy)$ and matching real and complex parts shows that
$$\Pi x = \rho(\cos(2\pi\varphi)x - \sin(2\pi\varphi)y)$$
$$\Pi y = \rho(\sin(2\pi\varphi)x + \cos(2\pi\varphi)y)$$

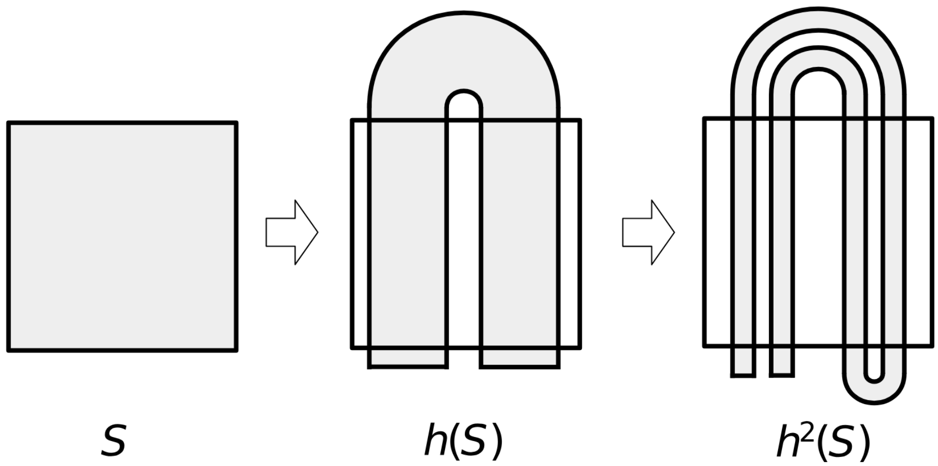
We can consider the sequences $\frac{\Pi^n \tilde{f}}{\rho^n}$ and $\frac{\Pi^n(p+\varepsilon\tilde{f})}{\varepsilon\rho^n}$. Both will describe the same trajectory and tend to the same plane as $n \to \infty$. We mentioned a perspective about Jordan blocks here, but I will omit it.
Finally, for uniqueness, suppose for contradiction that p_1 and p_2 are distinct invariant vectors. Since these vectors are invariant, their difference $d = p_1 - p_2$ will also be preserved under the transformation. From here, it is possible to show that there should exist an invariant vector $\tilde{p}$ on the boundary ∂P. Since $\tilde{p}$ is invariant, it should stay, but strict positivity of Π^m means that $\Pi^m \tilde{p} = \tilde{p}$ is not on the boundary, a contradiction. Hence $p_1 = p_2$. Also, it is possible to show the existence of a vector with eigenvalue 1, but we omit this.

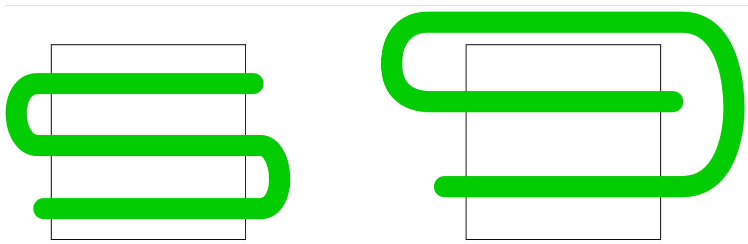
■

Remark. There is a deep connection between Fourier analysis and measure theory. Here, maximal eigenvalue corresponds to the eigenvector which represents invariant measure in Markov chains. Check video at 1:12:30 for more elaboration.

8.2 Smale horseshoe

Here are some visual representations of the Smale horseshoe, which describe a map of a rectangle into certain parts of itself.





The differences in these horseshoes depend on the orientation of the transformed figures. For now, we want to think about the the sections of the transformed rectangle which intersect with the original rectangle. These intersections will usually appear as two strips running vertically along the rectangle. One pattern of iteration we can do is to take these vertical strips (i.e. rectangles) and apply the horsehoe map again to them. Over many iterations, the intersections of these sets become a Cantor-like set. So, for all forward iterations (FH), we have an intersection like $K \times I$, where I is our interval. Meanwhile, for backward iterations ($F^{-1}H$), we have $I \times K$. Thus, for all iterations, we have $K \times K$. This product of Cantor sets is called the Smale horseshoe set. This turns out to be the most typical case for dynamical systems (i.e. for arbitrary dynamical systems in dimension greater than 1). We can characterize elements $x \in (K \times K)$ by their Fate map.

9 March 12, 2025

9.1 More Notions of Measure on Dynamical Systems

Definition. *Mixing*
A map is *mixing* if, given an invariant measure μ, for all A, B, the following holds

$$\lim_{n\to\infty} \mu(f^{-n}(A) \cap B) = \mu(A)\mu(B)$$

Note that f will usually be a diffeomorphism, but it is not required to be one.

What is the intuitive idea behind this definition? Given two arbitrary sets A and B, we want to think of B as some neighborhood, and as we iterate over A, our set will become more and more equidistributed over our phase space. Hence, it will be in the set B in the same proportion relative to the how much of the phase space it occupies. The main phenomenon we are capturing is this equidistribution over the phase space as n increases.
As an example, rotations are not mixing. On the other hand, Anosov diffeomorphisms are mixing with respect to the Lebesgue measure.

Proposition. Mixing implies ergodicity.

Recall that a measure-preserving dynamical system is ergodic if every invariant measurable set has measure zero or measure one.
Proof. Assume that B is an invariant set for our map f. Using our definition for mixing, we can put $A = B$. Now, since f is mixing, we may deduce

$$\mu(B) = \mu(f^{-n}(B) \cap B) = \mu(B)\mu(B)$$

where the first equality comes from the invariance of the set B. Hence, we conclude that

$$\mu(B) = \mu(B)^2$$

Of course, there are only two solutions to this equation, $\mu(B) = 0$ or $\mu(B) = 1$. Therefore, this map f is ergodic.

■

The above proposition shows that mixing is a stronger property than ergodicity.

9.2 Poincaré Recurrence Theorem and its Applications

Moving on, we discuss why ergodic properties of maps and measures are interesting, and how they connect to our previous discussion of ω-limit sets.

Theorem. *Poincaré Recurrence Theorem*
Let μ be a Borel, probability measure on X that is invariant under the map $f : X \to X$. Let A be any measurable set. For all $N \in \mathbb{N}$, we have

$$\mu\left(\{x \in A : \{f^n(x)\}_{n \geqslant N} \subseteq X \backslash A\}\right) = 0$$

In words, the measure of the orbit of x after some N that is outside of A has measure zero for all N (i.e. almost all points will return to A). We will denote $A_N = \{f^n(x)\}_{n \geqslant N}$.

Proof. It is sufficient to prove this just for $N = 1$. Recall

$$A_1 = \{x \in A : \{f^n(x)\}_{n \geqslant 1} \subseteq X \backslash A\}$$

We see that A_1 is measurable, since it consists of points $x \in A$ such that $x \in f^{-n}(X \backslash A)$ for all $n \in \mathbb{N}$. In other words, we may write

$$A_1 = A \cap \bigcap_{n \in \mathbb{N}} f^{-n}(X \backslash A)$$

Since A_1 is the intersection of measurable sets (preimages of measurable sets are themselves measurable), it is a measurable set. The set equality above follows directly from the definition of A_1.
Now, consider $f^{-1}(A_1) \subseteq X \backslash A$, but that the image of this set is in A. Hence

$$f^{-1}(A_1) \cap A_1 = \varnothing$$

Similarly, $f^{-2}(A_1) \subseteq X \backslash A$ so that $f^{-2}(A_1) \cap A_1 = \varnothing$. We may also conclude $f^{-2}(A_1) \cap f^{-1}(A_1) = \varnothing$. Why? So, we may see that

$$A_1, f^{-1}(A_1), \ldots, f^{-n}(A_1), \ldots$$

are mutually disjoint. Using countable additivity, we have

$$\mu\left(\bigcup_j f^{-j}(A_1)\right) = \sum_j \mu(f^{-j}(A_1)) = \sum_j \mu(A_1) < \infty$$

The last equality uses the invariance of the measure. We must have $\mu(A_1) = 0$, otherwise, the infinite sum of some constant finite number will be infinite. The result follows.

∎

Aside. We define a support. The *support* of a Borel measure μ on a topological space X is the smallest closed subset $C \subseteq X$ such that

$$\mu(X \backslash C) = 0$$

Equivalently, it is the closure of the set of all points $x \in X$ where every neighborhood U of x satisfies $\mu(U) > 0$.

$$\operatorname{supp}\mu = \overline{\{x \in X : \forall \text{ open } U \ni x, \mu(U) > 0\}}$$

Proposition. Let X be a compact manifold which is the phase space. Let μ be an ergodic measure with respect to f and $\operatorname{supp}\mu = X$. This means that every ball in the space with respect to the topology has nonzero measure. We may say that for almost all points $x \in X$, the orbit of x is dense in X. Equivalently, we may say that $\omega(x) = X$.

This proposition is interesting because it illustrates a connection between measure and topology.

Proof. Consider a base of the topology (i.e. the set of all possible neighborhoods). Since our set is a manifold, we can find a countable base, $\{U_i\}$. Since the support of μ is X, we have

$$\mu(U_i) > 0$$

for all U_i. First, we define an invariant set below

$$B_i = \{x \in X \mid (\forall N)(\exists n \geqslant N) : f^n(x) \in U_i\}$$

This is an invariant set because if we iterate it (remove a finite number of points from the beginning of the sequence), the resulting set of points will still have this property. Next, to apply the Poincaré recurrence theorem, we consider

$$A_i = B_i \cap U_i$$

This is the set of points in U_i which return to U_i infinitely many times. To this set, we may define $A_{i,N}$, the set of points which do not visit U_i after N iterations. Hence

$$A_i = U_i \setminus \left(\bigcup_n A_{i,n}\right)$$

Essentially, we consider all the points in U_i and remove those which do not return after a finite number of iterations. Observe that

$$\mu(A_i) \geqslant \mu(U_i) - \sum_n \mu(A_{i,n}) = \mu(U_i) > 0$$

where we have used $\sum_n \mu(A_{i,n}) = 0$, the result of the Poincaré recurrence theorem. We have $\mu(A_i) \leqslant \mu(U_i)$ by the monotonicity of measures, so that $\mu(A_i) = \mu(U_i)$. Further, since B_i contains A_i, we must have $\mu(B_i) > 0$. By the ergodicity of the measure and its invariance, we conclude that

$$\mu(B_i) = 1$$

Now, denote $B = \bigcap_i B_i$ and $\mu(B) = 1$. We have the property that $\text{orb}(x) \cap U_i \neq \varnothing$. Why? This means that the orbit of x intersects every neighborhood of X, which is precisely the definition of density in X. Since our set is a manifold, it may be demonstrated independently that we visit any neighborhood infinitely many times. From this, we may conclude $\omega(x) = X$.

■

Proposition. Recall that unique ergodicity means that there is only one invariant measure. Let $f : X \to X$ be uniquely ergodic. Then, for all $x \in \operatorname{supp} \mu$, we have $\omega(x) = \operatorname{supp} \mu$. This property is called *minimality*, and the support of μ is called the *minimal set*.

Proof. Let us start with some point $x \in X$. Consider $\omega(x)$, which is the set of all limit points of the orbit of x. Hence, we should have

$$\omega(x) \subseteq \overline{\text{orb}(x)} \subseteq \overline{\text{supp}(\mu)} = \operatorname{supp} \mu$$

The last equality holds since the support is always closed (by definition). By the Krylov-Bogolyubov theorem, any continuous map f on any compact set has invariant measure (recall that from previous lectures, are maps are always continuous). Consider the map

$$f\big|_{\omega(x)} : \omega(x) \to \omega(x)$$

Since this is a continuous map on a compact set, there exists an invariant measure ν. We may extend this measure to the entire phase space by taking

$$\nu(A) = \nu(A \cap \omega(x))$$

for $A \subseteq X$. It is easy to see that this extended measure will still be Borel. So, ν now is an invariant, Borel measure of f. By the unique ergodicity of f, we may conclude that $\mu = \nu$. In particular, we have $\operatorname{supp} \nu = \operatorname{supp} \mu$. But we have $\operatorname{supp} \nu \subseteq \omega(x)$, since we initially defined it on the restriction of f to $\omega(x)$. However, we have shown previously that $\omega(x) \subseteq \operatorname{supp} \mu$. Therefore,

$$\omega(x) = \text{supp}(\mu)$$

■

So, given a Borel measure and a uniquely ergodic map, the previous two propositions together show that all orbits are dense. In practice, this happens very rarely.

9.3 More on Linear Anosov Maps

Theorem. *Linear Anosov Map is Mixing*
The map $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ on the torus $\mathbb{T}^2$ is mixing with respect to the Lebesgue measure (which is invariant, since the determinant of the map is 1).

Proof. Consider the torus $\mathbb{T}^2$, represented by a square with its opposite edges identified. We may choose two eigenvalues which are orthogonal (since the matrix is symmetric). We will have two types of rectangles:

1. The rectangle has its axes parallel to the eigenvalues of our map (the Anosov diffeomorphism).
2. The rectangle will have its axes parallel to sides of the torus.

To show mixing, we must show

$$\mu(f^{-n}(A) \cap B) \xrightarrow[n\to\infty]{} \mu(A)\mu(B)$$

Step 1. We will show this to be the case when A is a rectangle of the first type and B is a rectangle of the second type. First, observe that $f^{-n}(A)$ also belongs to rectangles of the first type, because the sides of the rectangle are eigenvalues. We also remark that $f^{-1}(A)$ will be very thin and long (in pictures, it will look like the trajectory of irrational flow on a torus). Let y be some point on the shorter side of the rectangle, and $J_{n,y}$ be a line segment parallel to the longer sides of the rectangle which passes through y (and stops at the opposite, short side). Then, our rectangle is equal to the set

$$\bigcup_y J_{n,y}$$

Similarly, we may define rectangles of the second type using a union of line segments by letting L_q be a horizontal line segment which intersects the point q on the side that rectangle. Let B be the intersection of the first rectangle with the second and call $B_{h,q}$ the segment $B \cap L_q$. We claim that

$$\frac{\#\{J_{n,y} \cap B_{h,q}\}}{\#\{J_{n,y} \cap L\}} \xrightarrow[n\to\infty]{} \mathrm{Leb}_1(B_{h,q})$$

Further, the convergence is uniform over all values of y. This holds because if we consider the trajectories of the rectangles of the first type, it appears as the irrational winding on a torus. Hence, these points of irrational rotation on L_q are equidistributed, which yields the above formula. Continuing, we have

$$\frac{\mu(J_{n,y} \cap B)}{\mu(J_{n,y})} \rightrightarrows \mathrm{Leb}_1(B_n)\,\mathrm{Leb}_1(b_v) = \mu(B)$$

where b_v indicates vertical. We have gone from the previous equation to the current one by integration with respect to q. Now, we want to integrate with respect to y to get

$$\frac{\mu(f^{-n}(A)\cap B)}{\mu(f^{-n}(A))} \xrightarrow[n\to\infty]{} \mu(B)$$

But since the Lebesgue measure is invariant, we have $\mu(f^{-n}(A)) = \mu(A)$. From this, we obtain our desired equality

$$\mu(f^{-n}(A)\cap B) \xrightarrow[n\to\infty]{} \mu(A)\mu(B)$$

Step 2. Let $A = \bigsqcup_{i=1}^{k} A_i$ be a disjoint union of rectangles of the first type and $B = \bigsqcup_{j=1}^{m} B_j$ is a disjoint union of rectangles of the second type.
Since our map is a diffeomorphism, the preimages of these disjoint sets will also be disjoint.

$$\mu(f^{-n}(A)\cap B) = \sum_i\sum_j \mu(f^{-n}(A_i)\cap B_j) = \sum_i\sum_j \mu(A_i)\mu(B_j) = \mu(A)\mu(B)$$

where the penultimate equality uses the first step.
Step 3. Let $\varepsilon > 0$ and let $\tilde{A},\tilde{B}$ are arbitrary measurable sets. Recall that the Lebesgue measure has the property that

$$\mu(A\Delta\tilde{A}) < \frac{1}{4}\varepsilon,\ \ \mu(B\Delta\tilde{B}) < \frac{1}{4}\varepsilon$$

where Δ indicates the symmetric difference. Let us inspect the difference

$$|\mu(f^{-n}(\tilde{A})\cap\tilde{B}) - \mu(\tilde{A})\mu(\tilde{B})|$$

Observe that

$$\begin{aligned}\mu(\tilde{A})\mu(\tilde{B}) &\leqslant \mu(A)\mu(\tilde{B}) + \mu(A\Delta\tilde{A})\mu(\tilde{B})\\ &\leqslant \mu(A)\mu(B) + \mu(A\Delta\tilde{A})\cdot 1 + \mu(A)\mu(B\Delta\tilde{B})\\ &\leqslant \mu(A)\mu(B) + \frac{\varepsilon}{2}\end{aligned}$$

An analogous argument shows $\mu(A)\mu(B) - \frac{\varepsilon}{2} \leqslant \mu(\tilde{A})\mu(\tilde{B})$. Now, we have

$$\begin{aligned}\mu(f^{-n}(\tilde{A})\cap\tilde{B}) &\leqslant \mu(f^{-n}(A)\cap B) + \mu(f^{-n}(A\Delta\tilde{A})\cap B)\\ &\leqslant \mu(f^{-n}(A\Delta\tilde{A})) = \mu(A\Delta\tilde{A}) < \frac{\varepsilon}{4}\end{aligned}$$

where we have used the invariance of the measure in the last inequality. It may be shown that

$$\mu(f^{-n}(A)\cap B) - \frac{\varepsilon}{2} \leqslant \mu(f^{-n}(\tilde{A})\cap\tilde{B}) \leqslant \mu(f^{-n}(A)\cap B) + \frac{\varepsilon}{2}$$

Returning to the original difference we were considering, we may achieve the estimate

$$|\mu(f^{-n}(\tilde{A})\cap\tilde{B}) - \mu(\tilde{A})\mu(\tilde{B})| \leqslant \varepsilon + |\mu(f^{-n}(A)\cap B) - \mu(A)\mu(B)|$$

Finally, since ε is was arbitrary and since we have shown in the previous step that

$$|\mu(f^{-n}(A) \cap B) - \mu(A)\mu(B)| \xrightarrow[n\to\infty]{} 0$$

the same conclusion follows for arbitrary $\tilde{A}$ and $\tilde{B}$ by the inequality.

■

Corollary. In particular, the Lebesgue measure is ergodic, and almost all orbits are dense.

Remark. This corollary holds in spite of the fact that there are a countable number of periodic points which are dense (points with dense orbits are themselves dense?) and in fact have full Lebesgue measure.

9.4 Exercises

1. Show that rotations are not mixing.

10 March 12, 2025

10.1 Ergodicity implies Extremality

This lecture will focus on properties of measures in general, with particular attention to **ergodic** measures. Recall that an ergodic measure is a measure such that any invariant set of that measure has measure 0 or measure 1. Another property is being **extremal**. Extremal measures are of the form $\alpha\mu_1 + (1-\alpha)\mu_2$, where the coefficients may be generalized to values of some convex hull when we have more than two invariant measures. This sum of measures will be again invariant. The measure is extremal when $\alpha = 0$ or $\alpha = 1$.
We demonstrated previously that if our measure is extremal, then it is ergodic. Now, if our measure is ergodic, can we say anything about if it is extremal? For this, we discuss theorem relating to measure decomposition.

Theorem. *Measure Decomposition*
Let λ, μ be two Borelian, probabilistic measures. There exist s_μ, a_μ (where s and a are chosen to denote singularity and absolute continuity) and there exists a number $\alpha \in [0, 1]$ for which the unique decomposition below holds.

$$\lambda = \alpha s_\mu + (1-\alpha) a_\mu$$

To be explicit, α_μ that is absolutely continuous with respect to μ. Here, absolute continuity means

$$\int_X \varphi \, d\alpha_\mu = \int \varphi\rho \, d\mu$$

The function φ is arbitrary. The letter ρ represents the density function. There is a density of one measure with respect to another, which means that there is a function which does not depend on φ such that the equality holds for all continuous φ.
Further s_μ is mutually singular with μ. This means that there exists a set $A \subseteq X$ such that $\mu(A) = 0$ but $s_\mu(A) = 1$.

Proposition. If μ is ergodic, then μ is extremal. To be explicit, μ is ergodic with respect to some function f and it is extremal in the set of invariant measures.

Proof. Suppose that μ is ergodic and not extremal. It follows that we may represent μ as the sum of two invariant measures

$$\mu = \beta\lambda_1 + (1-\beta)\lambda_2$$

where β is not 0 or 1. Using the measure decomposition theorem, we may write

$$\lambda_1 = \alpha s_\mu + (1-\alpha) a_\mu$$

Since s_μ is mutually singular, there exists A such that $\mu(A) = 0$, $s_\mu(A) = 1$. We have

$$0 = \mu(A) = \beta\lambda_1(A) + (1-\beta)\lambda_2(A) \geqslant \beta\lambda_1(A) \geqslant \beta\alpha s_\mu(A)$$

Above, we have just used substitution and ignored nonnegative terms in the sum to obtain the inequalities. If both β and α are nonzero, then we have a contradiction. Hence, we must have $\beta = 0$ or $\alpha = 0$. Suppose β is zero, then we have shown that μ is extremal, which is a contradiction. Suppose α is zero. In this case, we have $\lambda_1 = a_\mu$. We invoke the following lemma.

Lemma. Let $(X, \mathcal{B}, \mu)$ be a probability space, $f : X \to X$ a measurable map, and $\varphi : X \to \mathbb{R} \cup \{\pm\infty\}$ a measurable function such that

$$\varphi = \varphi \circ f \quad (\text{ i.e. } \varphi(f(x)) = \varphi(x) \text{ for } \mu\text{-a.e. } x).$$

If μ is ergodic for f (every f-invariant set has μ-measure 0 or 1), then φ is constant μ-a.e.

Proof of the Lemma. Define the set

$$M_a = \{x \in X : \varphi(x) \leqslant a\}$$

First, note that M_a is invariant. Let $x \in M_a$. Then $\varphi(f(x)) = \varphi(x) \leqslant a$. Hence $f(x) \in M_a$. Since M_a is invariant, we must have $\mu(A) = 0$ or $\mu(A) = 1$. We claim that it is not possible for all sets of this type to have either measure zero or measure one. Suppose that $\mu(M_a) = 0$ for every $a \in \mathbb{R}$. Observe that if $a \leqslant b$, then $M_a \subseteq M_b$. Taking a sequence $a_n \to +\infty$, we have

$$\bigcup_n M_{a_n} = \{x : \varphi(x) < +\infty\}$$

By countable sub-additivity,

$$\mu(\{\varphi < \infty\}) = \mu\left(\bigcup_{n=1}^{\infty} M_{a_n}\right) \leqslant \sum_{n=1}^{\infty} \mu\left(M_{a_n}\right) = 0$$

This means that all the set of points with finite value has measure zero, so that everywhere else, we have $\varphi(x) = +\infty$, which is a constant. If $\mu(M_a) = 1$ for every $a \in \mathbb{R}$, we may use nestedness in the other direction to see that

$$\mu\left(\bigcup_n M_{a_n}^c\right) \leqslant \sum_{n=1}^{\infty} \mu\left(M_{a_n}^c\right) = \sum_{n=1}^{\infty} 0 = 0$$

From this, we may conclude $\mu(\{\varphi = -\infty\}) = 1$

Hence, we may denote

$$a_1 = \sup\{a : \mu(M_a) = 0\}, \;\; a_2 = \inf\{a : \mu(M_a) = 1\}$$

By the monotonicity of the map $a \mapsto \mu(M_a)$, we see that $a_1 \leqslant a_2$. Suppose, for contradiction, that there exists a c so that $a_1 < c < a_2$. Then, by definition we have $\mu(M_c) = 0$ and $\mu(M_c) = 1$, which is impossible. Call $a^* = a_1 = a_2$. It is possible to show that $\mu(\{\varphi \geqslant a^*\}) = 1$ and $\mu(\{\varphi \leqslant a^*\}) = 1$ so that their intersection is equal to 1. This means that our function is indeed almost everywhere constant, with the value a^*. □

Returning to our proof, we have that $\lambda_1 = a_\mu$, that is λ_1 is absolutely continuous to a_μ. This means that

$$\int_X \psi \, da_\mu = \int_X \psi \, \rho d\mu$$
$$\int_X \psi \circ f \, da_\mu = \int \psi \circ f \rho \, d\mu$$

Since our measure is invariant, the change of coordinates under the differential does not change our measure, so

$$\int_X \psi \circ f \, da_\mu = \int_X \psi \circ f \, \rho \circ f \, d\mu$$

From the equalities, we may conclude that $\rho = \rho \circ f$ almost everywhere with respect to μ. By the lemma we just proved, ρ is constant almost everywhere with respect to μ. We may now write

$$\lambda_1 = c\mu = \mu$$

where we know that $c = 1$ because both measures are probabilistic. Therefore, $\beta = 1$ so that μ is extremal, a contradiction.

■

10.2 The Birkhoff Ergodic Theorem

We turn our discussion to one of the most brilliant results of dynamical systems with connections to physics, mechanics, topology, PDEs, and even more.

Let $f : X \to X$ be a homeomorphism and let μ be a Borel, invariant probability measure. Let $\varphi : X \to \mathbb{R}$ be a measurable function and $\varphi \in L^1(X, \mu)$. Define

$$A_n(\varphi, x) = \frac{\varphi(x) + \ldots + \varphi(f^n(x))}{n+1}$$

The time average is defined as

$$\lim_{n\to\infty} A_n(\varphi, x) = A_\infty(\varphi, x)$$

The space average is defined as

$$\int \varphi \, d\mu = S$$

There are two formulations of the ergodic theorem. One is more general and the second is more relevant for applications.

Theorem. *Birkhoff Ergodic Theorem*
This is the pointwise version of the theorem. All notation is defined above.

1. (Pointwise convergence.) There exists a measurable function

$$A_\infty(x) = \lim_{n\to\infty} A_n(\varphi, x)$$

for μ-almost every x.

2. (Invariance and integrals.) That limit A_∞ satisfies

$$A_\infty \circ f = A_\infty \quad \text{and} \quad \int A_\infty \, d\mu = \int \varphi \, d\mu$$

for μ-almost every x.

3. (Ergodic case.) If, in addition, μ is ergodic for f (meaning any f-invariant measurable set has μ-measure 0 or 1), then

$$A_\infty(x) = \int \varphi \, d\mu \quad \text{for } \mu\text{-almost every } x$$

Remark. The first statement may appear strange because there is no measure in the definition of A_n, but measure is mentioned in the conclusion. However, this is alright; the claim is about the set that this equality holds on.

Proof. We proceed in steps.

Step 1. Let $a < b$. Define the set

$$B_{a,b} = \{x : \limsup A_n(\varphi, \cdot) > b, \liminf A_n(\varphi, \cdot) < a\}$$

Aside. To make the construction more concrete, we might consider the toy function $A_n(\varphi, x) = (-1)^n \cdot x$ and take φ as some fixed parameter which is not used. Then, $\liminf A_n = -|x|$ and $\limsup A_n = |x|$. If we take a value such as $x = 2$, then 2 and -2 are the values of the limsup and liminf respectively. If we choose $a = -1$ and $b = 1$, then $2 \in B_{-1,1}$. The key idea is whenever the limsup and liminf are not equal, we can do this.

If $\limsup A_n \neq \liminf A_n$, then there exist $a, b \in \mathbb{Q}$ such that $x \in B_{a,b}$. Also, if $\mu(B_{a,b}) = 0$ for all $a, b \in \mathbb{Q}$, then $\mu(\bigcup_{a,b\in\mathbb{Q}} B_{a,b}) = 0$, by countable sub-additivity. But then notice that

$$\{x : \limsup A_n \neq \liminf A_n\} \subseteq \bigcup_{a<b} B_{a,b}$$

$$\mu(\{x : \limsup A_n \neq \liminf A_n\}) \leqslant \mu\left(\bigcup_{a<b} B_{a,b}\right) = 0$$

So, if we prove $\mu(B_{a,b}) = 0$, we may conclude that the limit superior is equal to the limit inferior almost everywhere.

Step 2. We want to prove two inequalities

$$\int_{B_{a,b}} \varphi \, d\mu \geqslant b\mu(B_{a,b}), \quad \int_{B_{a,b}} \varphi \, d\mu \leqslant a\mu(B_{a,b})$$

Using these two inequalities and the fact that $a < b$, we can show that the measure of $B_{a,b}$ is zero. Observe that

$$A_n(-\varphi, \cdot) = -A_n(\varphi, \cdot)$$
$$\limsup A_\infty(\varphi, \cdot) = -\liminf A_\infty(-\varphi, \cdot)$$

Assume that the inequality on the left is true. Then, we see that

$$\int_{B_{-b,-a}} -\varphi \, d\mu \geqslant (-a)\mu(B_{-b,-a}) \implies \int_{B_{a,b}} \varphi \, d\mu \leqslant a\mu(B_{a,b})$$

The above implication holds via substitution and noticing that $B_{-b,-a} = B_{a,b}$. So, we only need to show one inequality is true and we will have the other. Denote $\psi = \varphi - b$, we want to show the equivalent statement

$$\int_{B_{a,b}} \psi \, d\mu \geqslant 0$$

Step 3. Define the set

$$C_n = \{x \in B_{a,b} : \max_{j \leqslant n} A_j(\varphi, \cdot) > b\}$$

Since we are looking at the maximum of the first n terms in these sets, it is clear that $C_n \subseteq C_{n+1}$. Further, $\bigcup_{n=1}^{\infty} C_n = B_{a,b}$. Also, define the characteristic function χ_n which is 1 on C_n and 0 otherwise. Thus, there exists an integral

$$\int_{C_n} \psi \, d\mu = \int_{B_{a,b}} \psi \cdot \chi_n \, d\mu \xrightarrow[n\to\infty]{} \int_{B_{a,b}} \psi \, d\mu$$

where the limit holds due to a theorem by Lebesgue (monotone convergence theorem). Consider the following statement

$$(\forall n) \int_{C_n} \psi \, d\mu \geqslant 0 \implies \int_{B_{a,b}} \psi \, d\mu \geqslant 0$$

If we can show that the premise holds, then we are closer to our conclusion.

Step 4. We may reformulate C_n in terms of new functions ψ_n

$$C_n = \{x \in B_{a,b} : \max_{j \leqslant n} A_j(\psi, \cdot) > 0\}$$

We denote

$$\psi_0 \equiv 0$$

$$\psi_n(x) = (n+1)A_n(\psi, x), \quad \Psi_n = \max_{0 \leqslant j \leqslant n} \psi_j \geqslant 0$$

We rewrite

$$C_n = \{x \in B_{a,b} : \Psi_n(x) > 0\}$$

This is the same definition because all ψ_n correspond to some A_n, except ψ_0. But ψ_0 does not add any new x values since we have a strict inequality $\max_{j \leqslant n} A_j(\psi, \cdot) > 0$ anyways.

Step 5. For all $j \leqslant n-1$, we have $\Psi_n \geqslant \psi_j$. Thus, $\Psi_n \circ f \geqslant \psi_j \circ f$. Observe that

$$\Psi_n \circ f + \psi \geqslant \psi_j \circ f + \psi = \psi_{j+1}$$

So

$$\psi_n \circ f + \psi \geqslant \max_{1 \leqslant j \leqslant n} \psi_j$$

$$\psi \geqslant \max_{1 \leqslant j \leqslant n} \psi_j - \psi_n \circ f$$

Let $x \in C_n$, in this case, $\max \psi_j > 0$ so that

$$\Psi_n = \max_{0 \leqslant j \leqslant n} \psi_j = \max_{1 \leqslant j \leqslant n} \psi_j$$

Now, on C_n, we have the following property

$$\psi(x) \geqslant \Psi_n(x) - \Psi_n \circ f(x)$$

Step 6. We may write

$$\int_{C_n} \psi \, d\mu \geqslant \int_{C_n} \Psi_n \, d\mu - \int \Psi_n \circ f \, d\mu$$

By definition, we know that

$$\int_{C_n} \Psi_n \, d\mu = \int_{B_{a,b}} \Psi_n \, d\mu$$

This is because C_n is the set where our function ψ is greater than 0, so $\Psi_n(x) = 0$ for $x \in B_{a,b} \backslash C_n$. Therefore

$$\int_{C_n} \psi \, d\mu \geqslant \int_{B_{a,b}} \Psi_n \, d\mu - \int_{B_{a,b}} \Psi_n \circ f \, d(\mu \circ f) = 0$$

In principle, when we perform a change of variables, we should change the set over which we integrate, but since our set is invariant $f^{-1}(B_{a,b}) = B_{a,b}$, we leave it. So, the inequality above is the proof of what we wanted, $\int_{C_n} \psi \, d\mu \geqslant 0$. ■

Aside. Why is the set $B_{a,b}$ invariant in the above proof? To see this, we need to check the definition. Recall

$$A_n(\varphi, x) = \frac{\varphi(x) + \ldots + \varphi(f^n(x))}{n+1}$$
$$A_n(\varphi, f(x)) = \frac{\varphi(f(x)) + \ldots + \varphi(f^{n+1}(x))}{n+1}$$

Notice that the difference is

$$A_n(\varphi, x) - A_n(\varphi, f(x)) = \frac{\varphi(x) - \varphi(f^{n+1}(x))}{n+1}$$

Since $|\varphi| \in L^1$, we have $\frac{|\varphi(f^n(x))|}{n}$ almost everywhere. So the difference tends to zero and $f(x) \in B_{a,b}$.

10.3 Some Applications of the Birkhoff Ergodic Theorem

We consider continued fractions. Notice that we may get a continued fraction representation of an improper fraction in the following way

$$\frac{19}{7} = 2 + \frac{5}{7} = 2 + \frac{1}{\frac{7}{5}} = 2 + \frac{1}{1 + \frac{2}{5}}$$
$$= 2 + \frac{1}{1 + \frac{1}{\frac{5}{2}}} = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{2}}}$$

This is a finite continued fraction. In fact for any real number, we have such a continued fraction representation. Gauss proved that hypergeometric functions may be represented as combinations of elementary functions based on how they work on continued fractions. Further, Lagrange also proved theorems on continued fractions, such as the theorem that if the continued fraction is periodic, then the number is quadratic irrational.
The Gauss map $g : [0,1] \to [0,1]$ is the following map:

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \\ \left\{\frac{1}{x}\right\} = \frac{1}{x} \mod 1 & \text{if } 0 < x \leqslant 1 \end{cases}$$

Here $\{x\}$ denotes the fractional part of x. We can write $\{x\} = x - [x]$ where $[x]$ is the integer part. Equivalently, $\{x\} = x \mod 1$. Remark that

$$\left[\frac{1}{x}\right] = n \quad \Leftrightarrow \quad n \leqslant \frac{1}{x} < n+1 \quad \Leftrightarrow \quad \frac{1}{n+1} < x \leqslant \frac{1}{n}$$

Using the Gauss map, we may write

$$x = \frac{1}{\left[\frac{1}{x}\right] + \frac{1}{\left[\frac{1}{g(x)}\right] + \frac{1}{\ldots}}}$$

One may ask about the distribution of the numbers $\left[\frac{1}{x}\right], \left[\frac{1}{g(x)}\right], \left[\frac{1}{g^2(x)}\right], \ldots$ The Gauss-Kuzmin theorem provides the answer about this. First, we give a setting. Take the Gauss map as above. For a continued fraction $[0; a_1, a_2, \ldots]$, we use the notation

$$a_n(x) = \left[\frac{1}{g^{n-1}(x)}\right], \quad n \geqslant 1$$

Define the measure

$$d\mu(x) = \frac{1}{\ln 2}\frac{dx}{1+x}, \quad x \in (0,1]$$

It turns out this measure is both g**-invariant and ergodic.** We now define the indicator of a digit. Fix $k \geqslant 1$ and set

$$I_k = \left(\frac{1}{k+1}, \frac{1}{k}\right], \quad \chi_k(x) = \mathbf{1}_{I_k}(x)$$

We have $a_n(x) = k \iff g^{n-1}(x) \in I_k$. The time-average of χ_k along the orbit of x is the same as the frequency of the digit k

$$A_n(\chi_k, x) := \frac{1}{n}\sum_{j=0}^{n-1} \chi_k(g^j(x)) = \frac{\#\{1 \leqslant j \leqslant n : a_j(x) = k\}}{n}$$

The Birkhoff ergodic theorem says that for every $f \in L^1(\mu)$

$$\lim_{n\to\infty} A_n(f, x) = \int_0^1 f\, d\mu$$

for μ-a.e. x. We may apply this conclusion for the bounded, integrable function $f = \chi_k$ to see that

$$\lim_{n\to\infty} A_n(\chi_k, x) = \int_0^1 \chi_k\, d\mu = \mu(I_k) = \frac{1}{\ln 2}\int_{I_k} \frac{dx}{1+x}$$

We compute

$$\mu\left(I_k\right) = \frac{1}{\ln 2}[\ln(1+x)]_{1/(k+1)}^{1/k} = \log_2\left(\frac{(k+1)^2}{k(k+2)}\right)$$

From the discussion above, we have the Gauss-Kuzmin theorem.

Theorem. *Gauss-Kuzmin*
Let g be the Gauss map and let $a_n(x)$ be the n^{th} continued-fraction digit of $x \in (0,1]$. Then for μ-almost every x (hence for Lebesgue-a.e. x, because μ has a continuous, positive density) and for every integer $k \geqslant 1$

$$\lim_{n\to\infty} \frac{\#\left\{1 \leqslant j \leqslant n : a_j(x) = k\right\}}{n} = \log_2\left(\frac{(k+1)^2}{k(k+2)}\right)$$

i.e. the asymptotic frequency of the digit k equals the Gauss-Kuzmin probability $P\left(a_1 = k\right) = \mu\left(I_k\right) = \log_2\left(\frac{(k+1)^2}{k(k+2)}\right)$.

Remark. The ergodicity of (g, μ) supplies the crucial hypothesis of the Birkhoff theorem; Birkhoff then converts the space average $\mu(I_k)$ into the time average that appears on the left, yielding the desired limit for almost every point.

11 March 19, 2025

11.1 Koopman Operator

We start with a dynamical system and let the evolution function $f : M \to M$ be a homeomorphism. Let $\varphi : M \to \mathbb{R}$ and $\varphi \in \mathscr{C}^0(M)$, the set of continuous functions on M.

Definition. *Koopman Operator*
The Koopman operator $K : \mathscr{C}^0(M) \to \mathscr{C}^0(M)$ is given by

$$K(\varphi) = \varphi \circ f$$

that is, $K\varphi(x) = \varphi(f(x))$. We call K the Koopman operator of f.

Notice that the Koopman operator K is linear, due to how addition and multiplication on function spaces are defined.

$$K(a\varphi + b\psi)(x) = (a\varphi + b\psi)(f(x)) = aK\varphi(x) + bK\psi(x)$$

Example. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(x_1, x_2) = (ax_1, bx_2 + (b - a^2)x_1^2)$$

Here is a nonlinear dynamical system. There are two invariant manifolds for this map. They are the sets $\{x_1 = 0\}$ and $\{x_2 = -x_1^2\}$. This suggests that for our Koopman operator, there should be two eigenfunctions: $(x_1, x_2) \mapsto x_1$ and $(x_1, x_2) \mapsto x_2 + x_1^2$, with corresponding eigenvalues a and b. Now, define the functions

$$\xi(x_1, x_2) = (x_1, x_2 + x_1^2, -x_1^2)$$

$$\xi(x_1, x_2) \xrightarrow{F} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & a^2 \end{pmatrix} \xi(x_1, x_2)$$

Further, taking the right side of the above map equal to $(\tilde{x_1}, \tilde{x_2})$, we may convert back via $\xi^{-1}(\tilde{x_1}, \tilde{x_2})$.
Looking at concrete values, consider $(x_1, x_2) = (1, 1)$. Applying F to $\xi(1, 1)$, we have

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \xrightarrow{F} \begin{pmatrix} a \\ 2b \\ -a^2 \end{pmatrix} \xrightarrow{F} \begin{pmatrix} a^2 \\ 2b^2 \\ a^4 \end{pmatrix}$$

According to our assumptions, there are two numbers corresponding to these lines, which are $(a^2, 2b^2 - a^4)$. It is easy to verify that ξ^{-1} maps this back to $(a^2, 2b^2, a^4)$. To summarize everything in a commutative diagram

$$\begin{array}{ccccc}
\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} & \xrightarrow{F} & \begin{pmatrix} a \\ 2b \\ -a^2 \end{pmatrix} & \xrightarrow{F} & \begin{pmatrix} a^2 \\ 2b^2 \\ a^4 \end{pmatrix} \\
\downarrow \xi^{-1} & & \downarrow \xi^{-1} & & \downarrow \xi^{-1} \\
(1,1) & \xrightarrow{f} & (a, 2b - a^2) & \xrightarrow{f} & (a^2, 2b^2 - a^4)
\end{array}$$

Note that there is a difference between the linear functions represented by matrices and vectors. So, what is the point of this construction. Consider how we have acted according to the following steps

1. We began with a nonlinear map f on the plane.
2. We find the eigenfunctions of the Koopman operator.
3. Then we use these eigenfunctions to construct a linear operator in higher dimensions, F.
4. Now, we are able to iterate F, which is a numerically cheap operation, and convert back in the end.

The point is we have reduced our nonlinear dynamic to a linear one. This is another view on the question of linearization of dynamical systems.

11.2 Measure and the Koopman Operator

Let M be our space and μ a measure on this space. There are two norms for functions

$$\|\varphi\|_{1,\mu} = \int_M |\varphi| \, d\mu$$
$$\langle \varphi, \psi \rangle_\mu = \int_M \varphi \cdot \psi \, d\mu$$

Proposition. μ is an invariant measure if and only if the Koopman operator is isometric, that is, for all φ we have

$$\|K\varphi\|_1 = \|\varphi\|_1$$

Note that this statement does not depend on the norm. It turns out this proposition follows from a change of variables in the integral formula.

Proof. Assume K is an isometry. Let $\varphi \geqslant 0$. We have

$$\|K\varphi\| = \int_M \varphi \circ f \, d\mu = \int_M \varphi \, d\mu = \int_M (\varphi \circ f) \, d(\mu \circ f)$$

Hence, comparing the second and last expression in the chain of equality above, $\mu = \mu \circ f$, showing that μ is invariant. On the other hand, suppose that μ is invariant. We have a similar chain of equality

$$\int \varphi \circ f \, d\mu = \int \varphi \circ f \, d(\mu \circ f) = \int \varphi \, d\mu$$

where the first equality follows by invariance and the second follows from a change of variables. Hence $\|K\varphi\| = \|\varphi\|$.

■

Theorem. *Koopman Theorem*
Let μ be an ergodic, Borelian, probabilistic measure. We have the following statements

1. $K\varphi = \lambda\varphi \implies |\lambda| = 1$, $|\varphi| =$ const. almost everywhere.
2. Eigenvalues are simple.
   This means if we have $K\varphi = \lambda\varphi$ and $K\psi = \lambda\psi$, then $\varphi = k\psi$, that is φ is proportional to ψ. So, there are no two eigenvectors with the same eigenvalue.
3. The set of eigenvalues of K define a subgroup on the unit circle (where the operation is multiplication).
4. If μ is mixing, then there exists a unique eigenvalue, 1.

Proof. Suppose that $K\varphi = \lambda\varphi$. Recall that ergodicity applies to sets which are invariant under the relevant transformation. From the previous proposition, μ is an invariant measure if and only if the Koopman operator is isometric. Since K is an isometry, we see that

$$\|K\varphi\| = \|\lambda\varphi\| \implies |\lambda| = 1$$

Since $\varphi \circ f = \lambda\varphi$, it follows that $|\varphi \circ f| = |\varphi|$, hence $|\varphi|$ is f-invariant. From a previous proposition, we know that for a function f, an ergodic measure μ, g invariant, we must have g be constant almost everywhere. Applying this proposition, we see that $|\varphi|$ is a constant almost everywhere. (Finish later. Class 11, 34:00)

11.3 The Perron-Frobenius Operator

The Perron-Frobenius a dual operator: it is an operator on measures which is the dual to the Koopman operator on functions. We define this object below.

Definition. *Perron-Frobenius Operator*
The Perron-Frobenius operator is denoted $P\mu$ and acts on a measure μ and it satisfies

$$\int \varphi \, dP\mu = \int (\varphi \circ f) \, d\mu$$

for all φ.

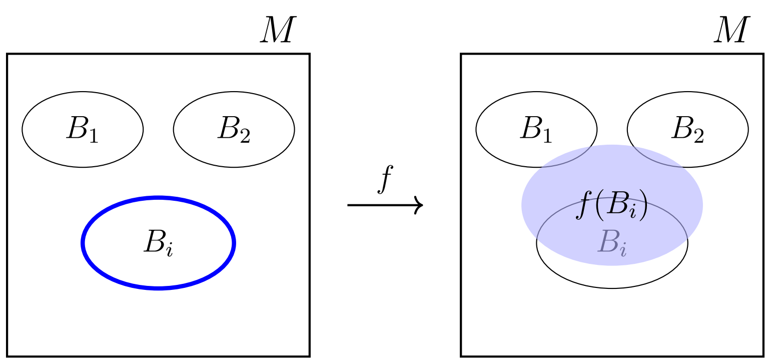
Remark. The relationship of this object to the Koopman operator should be obvious since

$$\int (\varphi \circ f) \, d\mu = \int (K\varphi) \, d\mu$$

This operator has become popular in recent decades because it may be applied to stochastic systems. One reason for our discussion of the Perron-Frobenius operator is because of its easier usage in introducing approximation.

11.4 Ulam approximation of Perron-Frobenius

Let our space M be split into regions $B_1, B_2, B_3, \ldots$ The intuitive idea is we have our evolution function f and some region B_i. Taking the image $f(B_i)$, there will be some of the original regions which intersect with this image. We would like to think of how many points in the image intersect with a particular region, say B_j.



We may use a matrix to describe these points of intersection. We define an operator $A = (a_{ij})$ which is an $N \times N$ matrix. Every element of the matrix is defined as follows

$$a_{ij} = \frac{\mathrm{Leb}(f(B_i) \cap B_j)}{\mathrm{Leb}\, f(B_i)}$$

We are using the Lebesgue measure, but in principle we may use any appropriate measure. What a_{ij} represents is the probability we are in B_j after one step, given that we started in B_i. Of course, the sum of these probabilities will be 1, since we must end up in one of the regions. So, A is a stochastic matrix as $\sum_j a_{ij} = 1$. We may define a probability measure by normalizing

$$\mu_i = \frac{\chi_{B_i} \cdot \mathrm{Leb}(B_i)}{\mathrm{Leb}(B_i)}$$

Now, we may look at this as an operator on specific measures which sends a measure to the sum of other measures. Continue at 1:07:00 L11

12 March 20, 2025

12.1 Review

We review some concepts first. We spoke briefly of linear circle maps and circle doubling (or multiplication). The point is we start with an identification of the unit circle S^1 with the quotient space $\mathbb{R}/\mathbb{Z} = [0, 1)$. We consider maps of the form $\varphi \mapsto k\varphi$. One fact to note about these maps is that for some point in the image, there are k preimages. For instance, for $\varphi \mapsto 2\varphi$, the preimages of 0.4 are 0.2 and 0.7. To find preimages for some image element of $\varphi \mapsto k\varphi$, we can add keep adding 1 to the image element and divide by k.
We talk about topological conjugacy. We will not repeat the details here because there is an existing section in the notes.

12.2 The Bernoulli Shift and Bernoulli Measure

We begin with some space A, which may be a finite set like $\{0, 1\}$, or an infinite set like the real line, or even a manifold. We choose the letter A to symbolize that this is the *alphabet* of our shift. We may considered two-sided or one-sided sequences, contained in the spaces of sequences $A^{\mathbb{Z}}$ or $A^{\mathbb{N}}$ respectively. We may define corresponding two-sided and one-sided shifts, $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ and $\sigma : A^{\mathbb{N}} \to A^{\mathbb{N}}$. An example of a one-sided shift is just the left shift $(a_1, a_2, a_3, \ldots) \mapsto (a_2, a_3, \ldots)$. An example of a two-sided shift is where we take $f(n) = a_n$ and move all mapped elements 'down' by one, so $g(n) = a_{n+1}$. Recall that a one-sided left shift corresponds to a circle map, since multiplying by the base of the representation has the effect of shifting all digits to the left (and the units digit is thrown away since we are considering equivalence classes).
Now, we discuss the consequences of defining a topology on A. We need the concept of an elementary cylinder, which is a set of words which have fixed values at specific indices.

$$\overline{a} \in C \iff \{(a_j)_{j\in\mathbb{Z}} : a_i \in A_i, A_i \subseteq A, A_i \text{ is open}, i \in [-M, N]\}$$

Notice that, by this definition, a cylinder is some subset of $A^{\mathbb{Z}}$. As an example, let $A = \{0, 1\}$ and suppose that we require

$$\ldots {}_5[011011]_{10} \ldots$$

or we might write $a_5 = 0, a_6 = 1, \ldots, a_{10} = 1$ (any choice of topology on $\{0, 1\}$ works). We take cylinders to be open sets of $A^{\mathbb{Z}}$ in our proposed topology.

Definition. *Bernoulli Measure*
Let A be some space and ν a measure on this space. We define the *Bernoulli measure*

on $A^{\mathbb{Z}}$ for cylinders by the product

$$\mu(C) = \prod_{i=M}^{N} \nu(A_i)$$

Note that we will require ν to be Borelian and probabilistic, and that μ will be probabilistic too as a result. Moreover, the idea of taking a product comes from the sense of a longer cylinder imposing more conditions on the sequences (which should correspond to a lower probability). It turns out that μ is σ-invariant with respect to $A^{\mathbb{Z}}$. Since $\sigma^{-1}(C) = \{\overline{a} : \sigma(\overline{a}) \in C\}$, it is just the same cylinder set with fixed indices shifted over to the right. Thus

$$\mu(C) = \mu(\sigma^{-1}(C))$$

We can say even more: σ is *mixing* with respect to μ (Bernoulli measures). Let us recall the definition of mixing. Given any two measurable sets A, B and an invariant measure μ, σ is mixing if the following holds

$$\lim_{n \to \infty} \mu(\sigma^{-n}(A) \cap B) = \mu(A)\mu(B)$$

Recall that we have already proved that linear Anosov diffeomorphisms are mixing. It turns out circle doubling is mixing as well.

Proposition. σ is *mixing* with respect to μ (Bernoulli measures).

Proof. Recall the basic proof strategy we used for Anosov diffeomorphisms:

1. Prove mixing for elementary sets. For linear Anosov diffeomorphisms, these are rectangles.
2. Prove this for disjoint union of elementary sets.
3. Approximate any set using disjoint unions.

We did not rely on any specific properties of linear Anosov diffeomorphisms in Steps 2 and 3 above, besides invertibility, in our previous proof. Hence, taking cylinders as our elementary sets, we may obtain an analogous proof, so long as we prove the first step holds. Q: Check this analogy more thoroughly. Something is not completely aligned since the previous proof relies on two types of rectangles.

We start with two cylinders A, B, defined by

$$A = \{\ldots [A_M \ldots A_N] \ldots\}$$
$$B = \{\ldots [B_P \ldots B_Q] \ldots\}$$

Let $N - n < P$, meaning $n > N - P$, to prevent overlaps, then, we have

$$\sigma^{-n}(A) \cap B = \{\dots [A_{M-n} \dots A_{N-n}] \dots [B_P \dots B_Q] \dots\}$$

Notice that this set is another cylinder, so that

$$\mu(\sigma^{-n}(A) \cap B) = \prod_{i=M-n}^{N-n} \nu(A_i) \cdot \prod_{i=P}^{Q} \nu(B_i) = \mu(A)\mu(B)$$

Observe that this is not only a limit, but when $n > N - P$, we have an equality.

■

Remark. In particular, σ is ergodic.

12.3 Some Connections to Probability Theory

Suppose that we have a standard six-sided die and assume that the results of rolling the die are independent and identically distributed (iid). So, let $X_1, \dots, X_n$ be iid random variables. Also, assume that $EX < \infty$. We may say

$$P\left(\lim_{n\to\infty} \frac{X_1 + \dots + X_n}{n} = EX\right) = 1$$

The statement that $\lim_{n\to\infty} \frac{X_1+\dots+X_n}{n} = EX$ with probability 1 called almost sure convergence. This statement above is true by the strong law of large numbers. We may describe concepts in probability in terms of measure. Let $X : \Omega \to \mathbb{R}$. Let our measure on Ω be $\tilde{\nu}$. Consider another set $\mathbb{R}^{\mathbb{N}}$. Let us now define another map $\pi : \Omega \to \mathbb{R}^{\mathbb{N}}$ by

$$\pi(\omega) = (X_1(\omega), X_2(\omega), \dots)$$

We define another Bernoulli shift $\sigma : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ which will just be a left shift. Define another measure

$$\nu = \tilde{\nu} \circ X_i^{-1}$$

Since our variables are iid, ν does not depend on the X_i we choose. So, ν will be a measure on $\mathbb{R}$. Finally, define a Bernoulli measure

$$\mu = \nu_1\nu_2\nu_3 \dots$$

Theorem. *Birkhoff Ergodic Theorem*
Let f be a dynamical system, μ be an ergodic measure, and $\varphi : M \to \mathbb{R}$ be an observable. Then, there is the following convergence

$$\lim_{n\to\infty} \frac{\varphi(x) + \varphi(f(x)) + \dots + \varphi(f^n(x))}{n+1} =_{\text{a.e.}} \int \varphi \, d\mu$$

We want to apply the Birkhoff Ergodic Theorem to the setup we have above. Let us take $\varphi(x) = x_1$, for $x \in \mathbb{R}^{\mathbb{N}}$, the shift σ as our f and the Bernoulli measure as μ, we have

$$\lim_{n\to\infty} \frac{x_1 + \ldots + x_n}{n} = \int_{\mathbb{R}^{\mathbb{N}}} x_1 \, d\mu = \int_{\mathbb{R}} x_1 \, d\nu = EX_1$$

Hence

$$\lim_{n\to\infty} \frac{X_1 + \ldots + X_n}{n} = EX$$

almost everywhere with respect to μ, and also with respect to $\tilde{\nu}$. Therefore, we have shown how the strong law of large numbers follows from the Birkhoff ergodic theorem. Q: Does it not matter when we consider one-sided or two-sided sequences here? Doesn't mixing only hold for two-sided sequences? In probability theory, there are two main theorems regarding averages, the strong law of large numbers and the central limit theorem.

Theorem. *Central Limit Theorem*
Using the same notation as above, we have

$$\frac{X_1 + \ldots + X_n - nEX_1}{\sqrt{n}} \rightrightarrows N(0, \sigma^2)$$

where $\rightrightarrows$ indicates convergence in distribution.

Remark. This probabilistic theorem explains why normal distribution is so popular and occurs in nature, blurring, noise, and so forth.
We talk about an analogue of the about theorem for Anosov diffeomorphisms. Previously, we have discussed linear Anosov diffeomorphisms, like $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. However, there is a more general notion of an Anosov diffeomorphism (i.e. nonlinear ones). Let $\varphi : \mathbb{T}^2 \to \mathbb{R}$ be a Hölder map (or Lipschitz, or $\mathscr{C}^1$). There is a no-cocycle condition. Let A be an Anosov diffeomorphism. For any φ (excluding some degenerate conditions), we have

$$\lim_{n\to\infty} \mu \left\{ x \in \mathbb{T}^2 \mid \frac{\sum_{i=0}^{n-1} \varphi(A^i x) - nE\varphi}{\sigma\sqrt{n}} < \alpha \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-\frac{1}{2}\mu^2} d\mu$$

where we note that $E\varphi = \int_{\mathbb{T}^2} \varphi \, d\mu$ and μ is some invariant measure (like the Lebesgue measure). This result is due to Bowen (1970s). So, there is a deep analogy between statistics and dynamics (hyperbolic dynamics).
We briefly touch on the maps of the Smale horseshoe (one that preserves and does not preserve orientation), and we illustrate how there is a topological conjugacy between Smale horseshoe and the Bernoulli shift.

13 March 26, 2025

13.1 Hyperbolicity

Recall that we have discussed the concept of periodic orbits. Suppose we have a fixed point which is a hyperbolic point, meaning that $|\lambda_i| < 1 < |\mu_j|$ for its eigenvalues λ_i, μ_j. It will be a saddle. This point is persistent, meaning that under perturbations of the map, we have periodic orbits of the same period and the same splitting of the tangent space (and the dynamics look similar). What about hyperbolicity in a more general context? We now go from hyperbolic points to hyperbolic sets

Definition. *Hyperbolic set*
Let $f : M \to M$ be a diffeomorphism. Let Λ be compact and invariant (i.e. $f(\Lambda) = \Lambda$). We call Λ a hyperbolic set if for all $x \in \Lambda$, we have

$$T_x M = E_x^u \oplus E_x^s$$

Note: These two distributions E_x^u, E_x^s by x are continuous with respect to x. This means that as we change our point slightly, the splitting still holds in that neighborhood. This point may be derived from the definition rather than stated.
We also require

$$Df_x(E_x^u) = E_{f(x)}^u \text{ and } Df_x(E_x^s) = E_{f(x)}^s$$

and there exists $C > 0$, $0 < \lambda, \mu < 1$ such that

$$\begin{aligned} \left\|Df^{-n}\xi\right\| &\leqslant C\lambda^n \left\|\xi\right\|, \forall \xi \in E_x^u \\ \left\|Df^{n}\eta\right\| &\leqslant C\mu^n \left\|\eta\right\|, \forall \eta \in E_x^s \end{aligned}$$

for natural numbers $n > 0$.

Remark. Intuitively, this definition is saying that a set is hyperbolic if for all points of the set, the tangent space at that point may be decomposed into two subspaces, an unstable and stable one, and where iterating in one subspace, we have a contraction, iterating in other direction gives us an expansion.

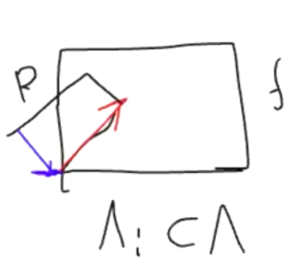
Proposition. f is an Anosov diffeomorphism if and only if M, the manifold, is a hyperbolic set.

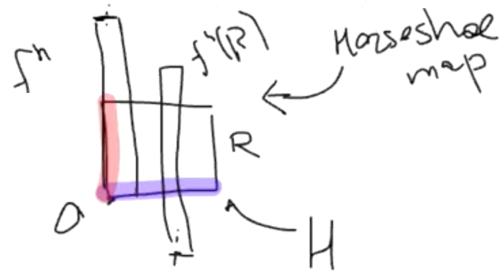
Examples. Create diagrams for these later.

1. As an example, we may consider the linear Anosov diffeomorphism $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ with the hyperbolic set Λ as $\mathbb{T}^2$.

2. Consider the Smale horseshoe and the horseshoe map F_H. We discussed previously the existence of a horseshoe set, the invariant set of F_H. This invariant set is $K \times K$, where K is the Cantor set. So $F_H(K \times K) = K \times K$. There is a distinction between vertical stripes of the original set and horizontal ones, where the horseshoe is homeomorphic for the horizontal, contracting direction. One issue with the Smale horseshoe is that it is not a manifold, so we *cannot* assert $T_xM = T_x\Lambda$.

Moving on, let us consider the torus and some square at the origin. We consider what happens to this shape under the contracting and expanding directions.



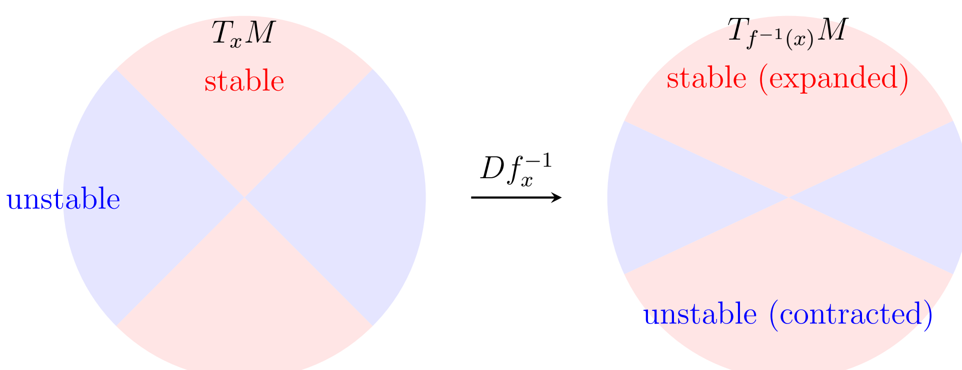


Definition. *Maximal hyperbolic set*
We call Λ a maximal hyperbolic set if

$$\Lambda \subseteq \tilde{\Lambda} \iff \Lambda = \tilde{\Lambda}$$

where $\tilde{\Lambda}$ is some other hyperbolic set. This maximal set always exists.

There are two main tools for the investigation of hyperbolic sets: *conic families* and *Markov partitions*. Below is a picture of a space that is split into its conic families and then what that splitting might look like in a backward iteration.



Definition. *Conic Family*
A *conic family* is a splitting T_xM where

$$Df_x(C_x^s) \subseteq C_{f(x)}^s \text{ and } D_xf^{-1}(C_x^u) \subseteq C_{f^{-1}(x)}^u$$

The notation C_x^s and C_x^u describe the stable and unstable conics, respectively. This is a useful tool because there are useful tools to obtain information from conics.

Example. We would like to locate the position of our stable and unstable subspaces, E_x^s, E_x^u. How may we determine this numerically. There is a trick called, "from the running start". We begin with the point x and a tangent space at x. We know that the tangent space may be decomposed as $E_x^u \oplus E_x^s$. To determine which conics are stable and unstable, we can consider the behavior of points in certain areas under the map (whether they are expanded or contracted). In our discussion, we consider the unstable direction and backward iterations of the cone. So, the unstable conics will be contracting (getting smaller and smaller in width) as we go backwards in time. There is some sense of a fixed point here if we take the intersection of all of these backward iterations; it turns out this intersection will be our E_x^u. A similar procedure may be carried out to determine E_x^s. Review this and draw a diagram from lecture.

Theorem.
Let Λ be a hyperbolic set for the diffeomorphism $f : M \to M$. There exists a neighborhood $V \supseteq \Lambda$ and $\varepsilon > 0$ such that if $\|f, g\|_{\mathscr{C}^1} < \varepsilon$, then

$$\Lambda_V^g = \bigcup_{n\in\mathbb{Z}} \overline{V}$$

is hyperbolic.

Remark. In practice, this statement means that if you have a hyperbolic set for some map, if you perturb this map slightly, you just perturb the hyperbolic set slightly, and the perturbed hyperbolic set is still hyperbolic.

Proof. Let our manifold M be Riemannian. The proof relies on vertical and horizontal conic fields. For the map, for any point x, we have the decomposition $E_x^s \oplus E_x^u$. Fix $\gamma > 0$. Define the object

$$H_x^\gamma = \{\xi + \eta \mid \xi \in E_x^s,\, \eta \in E_x^u,\, \|\eta\| \leqslant \gamma \|\xi\|\}$$

It may be shown that these above cones will be unstable cones.

$$V_x^\gamma = \{\xi + \eta \mid \xi \in E_x^s,\, \eta \in E_x^u,\, \|\eta\| \geqslant \gamma \|\xi\|\}$$

The splitting H_x^γ and V_x^γ defines a conic family. Observe that

$$Df_x H_x^\gamma \subseteq H_{f(x)}^{\lambda\mu^{-1}\gamma}$$
$$Df_x^{-1} V_x^\gamma \subseteq V_{f^{-1}(x)}^{\lambda\mu^{-1}\gamma}$$

where λ, μ come from the definition of hyperbolicity. Let $v \in H_x^\gamma$, then

$$\|Df_x v\| \geqslant \frac{\mu^{-1}}{1+\gamma}\|v\|$$

Let $w \in V_x^\gamma$, then

$$\|Df_x w\| \geqslant \frac{1+\gamma}{\lambda}\|w\|$$

Now, if we perturb our map, that is, taking g close enough to f, we have eigenvalues associated with g satisfying $0 < \tilde{\lambda}, \tilde{\mu} < 1$. Finish proof later, 1:28:00

13.2 Review of Some Problems

We discussed some strategies for Problems 4, 5, 6 in the set. We will omit this discussion and the reader may refer to the typed solutions.

14 March 26, 2025

14.1 Clarification and Review

At the beginning of the lecture, we clarified that the Markov partition is a distinct construction from a local rectangular section on the torus (or other manifold) which is transformed under some diffeomorphism in stable and unstable directions. Next, we give some elaboration on the meaning of conic in conic families. Let $\xi \in E^1$ and $\eta \in E^2$ be two vectors such that

$$\|\xi\| \geqslant C\|\eta\|$$

and where $E^1 + E^2 = \mathbb{R}^n$. In the language of "cones over a subspace" one usually calls

1. $\xi \in E^1$ the axis (or generating) direction of the cone,
2. $\eta \in E^2$ the base (or transverse) direction.

Geometrically, the cone itself is

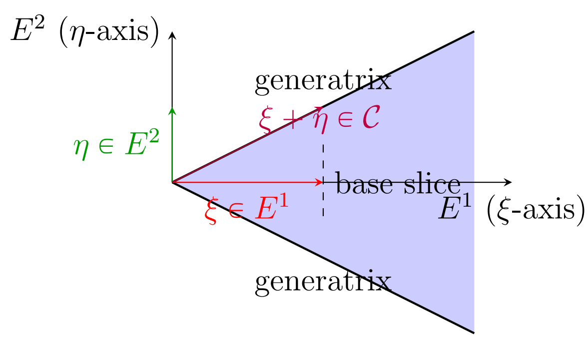
$$\mathcal{C} = \left\{\xi + \eta : \xi \in E^1, \eta \in E^2, \|\eta\| \leqslant \frac{1}{C}\|\xi\|\right\},$$

and its generatrices are the boundary rays

$$\left\{\xi + \eta : \|\eta\| = \frac{1}{C}\|\xi\|\right\}$$

while the base of the cone (at height 1 , say) is

$$\left\{\eta \in E^2 : \|\eta\| \leqslant \frac{1}{C}\right\}$$



14.2 Local Stable and Unstable Manifolds

Let Λ be a hyperbolic set and M a Riemannian manifold. For $x \in \Lambda$, we have

$$T_xM = E_x^u \oplus E_x^s$$

Another concept connected with hyperbolicity is that of local stable and local unstable manifolds, which are defined in the phase space with some conditions. In some circumstances we are able to pass from local to global manifolds.

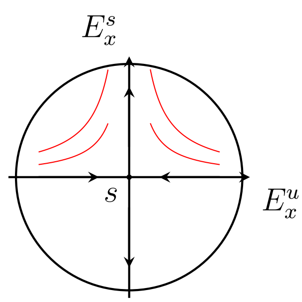
Definition. *Local Stable and Unstable Manifolds*
We define these local manifolds respectively. Note that there the parameter ε is implied from locality but sometimes not mentioned.

$$W^s_\varepsilon(x) = \{y \in M : \mathrm{dist}(f^n(x), f^n(y)) \leqslant \varepsilon, \forall n \geqslant 0\}$$
$$W^u_\varepsilon(x) = \{y \in M : \mathrm{dist}(f^{-n}(x), f^{-n}(y)) \leqslant \varepsilon, \forall n \geqslant 0\}$$

Remark. Here are some important observations.

1. The elements of the local stable and unstable manifold belong to your phase space and *not* the tangent space.
2. Although the name indicates that the object is a manifold, the proof of this is a non-trivial fact which follows from the Hadamard-Perron theorem. We will prove this later. The smoothness of the manifold relates to the smoothness of the dynamical system f.

Example. Let s be a fixed point, so that $f(s) = s$ and let it be a saddle as well. We also require $|\,\mathrm{sp}(Df_s)| \neq 1$. Below is a simplistic idea of the local situation.



Observe that the dynamics off of the axes follow a hyperbolic pattern (which one reason for the name), while the axes themselves represent the local stable and unstable manifolds. These local manifolds consist of the points which go towards the saddle in forward or reverse time.

Definition. *Global Stable and Unstable Manifolds*
These are just the union of local manifolds.

$$W^s(x) = \bigcup_{n=0}^{\infty} f^{-n}(W^s_\varepsilon(f^n(x)), \quad W^u(x) = \bigcup_{n=0}^{\infty} f^n(W^u_\varepsilon(f^{-n}(x))$$

It turns out that there exists some $\varepsilon > 0$ for which this definition holds.

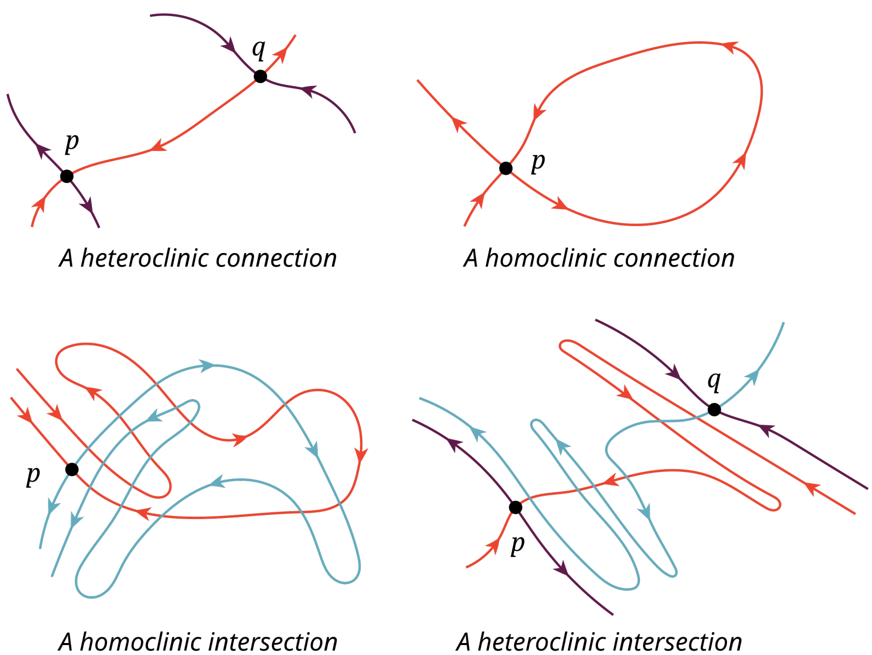
Definition. *Homoclinic Intersection*

A homoclinic intersection for a hyperbolic fixed (or periodic) point p is a point

$$q \neq p \quad \text{with} \quad q \in W^s(p) \cap W^u(p)$$

Equivalently, q lies on both the stable manifold (it approaches p under forward iteration) and the unstable manifold (it approaches p under backward iteration).

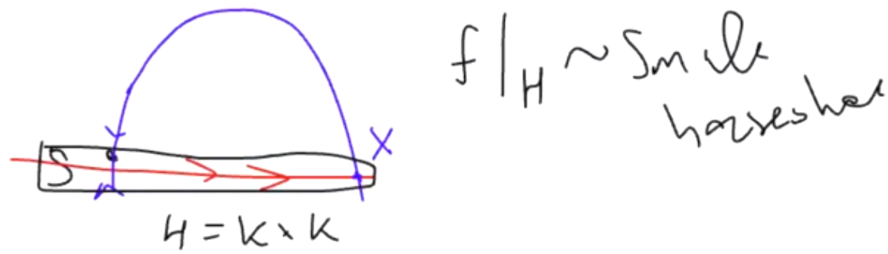
Remark. Such intersections are the "building blocks" of horseshoes and chaotic dynamics. For flows, there is no intersection of stable and unstable manifolds, they should coincide if they intersect. However, homoclinic intersections are possible for diffeomorphisms. Check why this is the case.



Remark. We discussed the figure of Poincaré's tangle to give more analysis for homoclinic intersections.

Remark. By iterating forwards and backwards on a rectangle R and looking at the intersection of the images and preimages, we come across the familiar construction of $K \times K$. See the section on Smale horseshoe for more detail.

The key idea of this discussion is that if we have a saddle s and a homoclinic intersection at some other point x, then there is a Smale horseshoe near the saddle, in some neighborhood. Some more context, background, and some additional visualizations are given in this document. Lastly, we make mention to the concept of homoclinic tangency and also briefly to the concept of universal dynamics. The presentation here was a bit unclear; review from 50:00



14.3 An Equivalent Condition for Structural Stability

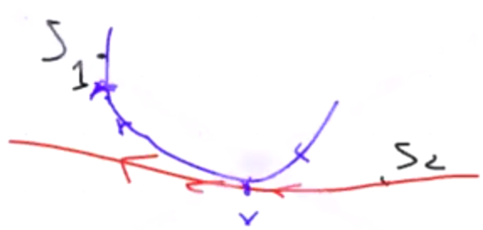
Recall the definition of structural stability. Let us consider $\mathscr{C}^1$ functions and $\mathscr{C}^1$-structural stability here. The function f is structurally stable if for all functions g in a neighborhood $U(f)$, we have g is topologically conjugated to f. We have the very important theorem.

Theorem. *Structural Stability Theorem*
This is sometimes called the Ω-stability theorem and it is variously attributed to Smale, Robinson, Palis, Mane, and others.
Let $f : M \to M$ be a $\mathscr{C}^1$-diffeomorphism of a compact manifold. Then f is $\mathscr{C}^1$-structurally stable if and only if it satisfies

1. *Axiom A*: the non-wandering set $\Omega(f)$ is hyperbolic and the periodic points are dense in $\Omega(f)$,
2. *Strong transversality*: all stable and unstable manifolds of points in $\Omega(f)$ meet each other transversely, and there are no heteroclinic cycles.

The second condition is sometimes called the no-cycles condition.

We elaborate on the two conditions above. First, we describe a heteroclinic tangency. Let s_1 and s_2 be two saddles. Consider a stable manifold of one of the saddles and an unstable manifold of another. If there is an intersection that is a point of tangency, then there is no strong transversality.



So, we may think of the absence of this picture above as strong transversality.

Next, we turn to Axiom A. We need to define a non-wandering set. Given a point x, how do we know whether or not it belongs to such a set?

Definition. *Non-wandering set*
Let $f : M \to M$ be a continuous map on a topological space (usually a compact manifold). A point $x \in M$ is called *non-wandering* for f if for every neighborhood U of x and every integer $N > 0$, there exists $n \geqslant N$ such that

$$f^n(U) \cap U \neq \varnothing$$

Equivalently, some iterate of the neighborhood U always returns and overlaps itself.

Remark. The non-wandering set depends on f, not any particular point in M.
We discuss some properties of non-wandering sets.

1. Assuming $x \in \omega(y)$, the set $\omega(y)$ is a subset of the non-wandering set. Why? Since $x \in \omega(y)$, there is a subsequence $f^{n_1}(y), f^{n_2}(y), f^{n_3}(y), \ldots$ in the orbit of y which converges to x. For any neighborhood U of x, it is possible to find two points $f^{n_a}(y), f^{n_b}(y)$ belonging to U, with $a < b$. This shows that $f^{n_b}(U) \cap U \neq \varnothing$, since $f^{n_a}(y) \in U$ will be mapped to $f^{n_b}(y) \in U$ after $n_b - n_a$ iterations.

2. The non-wandering set $\Omega(f)$ is closed and f-invariant.

3. Every periodic point lies in $\Omega(f)$. This follows from the first point, as periodic points belong to their ω-limit sets.

Our condition for Axiom A in the theorem may also correspond to the conditions

1. Periodic points are dense in the non-wandering set.

2. The non-wandering set is a disjoint union of hyperbolic sets.

Example. As a reminder, this theorem applies to diffeomorphisms. For instance, this does not work for circle doubling, but for linear Anosov diffeomorphisms, this works. To begin, we ask: what is the non-wandering set for the linear Anosov diffeomorphism? It turns out to be the entire torus $\mathbb{T}^2$. We saw previously that we have a dense orbit so that $\omega(x) = \mathbb{T}^2$. We know that rational points are periodic, so that they are dense in the non-wandering set. Further, linear Anosov diffeomorphism has a hyperbolic structure on all of the torus, so the non-wandering set is a trivial union of hyperbolic sets and is itself a hyperbolic set. Hence, it satisfies Axiom A.
Moreover, strong transversality is satisfied because stable and unstable manifolds meet transversely (no tangencies). Since the linear Anosov diffeomorphism satisfies these two properties, we know it is structurally stable from the theorem.

Remark. One difficulty that arises from employing this result is that many results in measure theory require $\mathscr{C}^2$ conditions rather than $\mathscr{C}^1$, so it is important to know alternate ways to arrive at results.

14.4 Smale Counterexample

Some descriptions of this may be found in an article by Smale and it is briefly mentioned in Katok and Hasselblatt's book. We discuss a naïve approach.

We consider the counterexample in some 3-dimensional manifold, say $\mathbb{S}^3$. The counterexample is made for the following conjecture: *Structurally stable maps are an open and dense set in $\mathscr{C}^1$, in $\mathscr{C}^2$.* Now, for flows in 2-dimensional spheres, this is true due to the Andronov-Pontryagin theorem, however, for higher dimensions it is not true.

To construct the counterexample, we start with with a 2-dimensional torus $\mathbb{T}^2$ and the linear Anosov diffeomorphism on this object. We consider a neighborhood of the torus, with the three dimensional coordinate (x, a), where $x \in \mathbb{T}^2$ and $a \in (-\varepsilon, \varepsilon)$. Taking the linear Anosov diffeomorphism as $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, we have the map

$$(x, a) \mapsto \left(Ax, \frac{a}{2}\right)$$

It may be shown that $\Lambda = \mathbb{T}^2 \times \{0\}$ is a hyperbolic set. So, we have stable and unstable manifolds. The unstable manifolds coincide with the unstable manifolds of the Anosov diffeomorphism. Meanwhile, stable manifolds will be 2-dimensional and go along the 1-dimensional stable manifold in the torus for the Anosov diffeomorphism. We may denote this stable manifold by

$$\operatorname{span}((v_s, 0), (0, 0, 1)) + (x, 0)$$

where v_s is a vector of the stable manifold in the torus. Now, in the sphere, we may imagine that there is another saddle and trajectory, which also has a 2-dimensional stable manifold and 1-dimensional unstable manifold. When we iterate along the unstable manifold, between x and $f(x)$, we will have something like a parabolic curve. When we iterate this map, we want to send this unstable manifold to the torus.

We now give an imprecise justification. Since these unstable manifolds are everywhere, there will be one which ends up tangent to the stable manifold we identified previously. So, what we have is a tangency between the unstable manifold of a saddle s and the stable manifold of x. We denote these objects $W^u(s)$ and $W^s(x)$ respectively. We define the set

$$\{x : W^u(s) \cap W^s(x) \neq \varnothing, W^u(s) \text{ one semispace w.r.t. } W^s(x)\}$$

to describe parabolic tangency. Importantly, for all $\tilde{x} \in U_\delta(x)$, there exists $\tilde{f}$ such that $\text{dist}(f, \tilde{f}) < \varepsilon$ so that there is a tangency into the stable manifold of $\tilde{x}$. But in a neighborhood of x, there are a dense set of periodic orbits, so that we may find two orbits with different periods. The key idea to obtain a contradiction at this step is to see that under a perturbation, we may obtain two maps which are topologically conjugated to each other, but which map periodic sets of different periods to each other. But under topological conjugacy, the period is invariant, so this is impossible.

15 April 9, 2025

15.1 Review

In the previous lecture, we introduced some criteria for structural stability.

1. It should satisfy the Axiom A property: we are hyperbolic on a non-wandering set.

2. Strong transversality.

We know that these criteria are not always fulfilled, moreover, there exist open domains in the set of diffeomorphisms which are not structurally stable. For instance, in Diffeo(M^3) there is the Smale counterexample.

15.2 Returning to Flows on $\mathbb{S}^2$

We recall and formulate again the Andronov-Pontryagin theorem. Note that the general case was proven by Peixoto, so sometimes it is referred to as Peixoto's theorem.

Theorem. *Andronov-Pontryagin Criteria*
Consider the conditions below for a vector field:

1. All fixed points are hyperbolic.

2. There are no saddle connections.

3. All periodic trajectories are limit cycles and hyperbolic.

We have the following statements

1. The set of vector fields satisfying these three conditions is an open and dense set in the set of all vector fields of $\mathbb{S}^2$.

2. Vector fields satisfying these three conditions are structurally stable.

Remark. In fact, this is true for any two-dimensional surface and not just spheres.
Before proceeding to the proof, we make a review of some concepts. We begin with linear fixed points. We call these fixed points for flows and singular points for vector fields. We start with a system that can generate a flow, and it will be linear, and two-dimensional. It is useful to think about the plane and the following linear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

Note that 0 is a fixed point of our flow. We want to understand how trajectories behave near this fixed point. To do so, we should recall a solution of this system. One way we can write such solutions is with matrix exponentiation

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{At}\begin{pmatrix} x_0, y_0 \end{pmatrix}$$

Another perspective involves considering a linear change of coordinates.

$$\dot{\tilde{x}} = \lambda_1 \tilde{x}, \quad \dot{\tilde{y}} = \lambda_2 \tilde{y}$$

In this case, we have a solution of the form

$$\tilde{x}(t) = e^{\lambda_1 t}\tilde{x}_0, \quad \tilde{y}(t) = e^{\lambda_2 t}\tilde{y}_0$$

Hence, we have a map of a flow

$$g^1 : \begin{cases} \tilde{x}_0 \mapsto e^{\lambda_1}\tilde{x}_0 \\ \tilde{y}_0 \mapsto e^{\lambda_2}\tilde{y}_0 \end{cases}$$

Now, we want to ask when the point $(0, 0)$ is hyperbolic. Recall that a hyperbolic point is a point where eigenvalues of the differential are not equal to 1. Referencing g^1 above, we see that the spectrum is $\mathrm{sp}(g^1) = \{e^{\lambda_1}, e^{\lambda_2}\}$. Based on the condition for hyperbolicity, we want $|e^{\lambda_1}| \neq 1$ and $|e^{\lambda_2}| \neq 1$. This occurs when λ_1 and λ_2 are not purely imaginary, that is, they are of the form $a \pm ib$ where $a \neq 0$. This is the condition of hyperbolicity for vector fields. To be explicit, since $|e^{ix}| = 1$, we have

$$\mathrm{Re}(\lambda_i) > 0 \implies |e^{\lambda_i}| > 1, \quad \mathrm{Re}(\lambda_i) < 0 \implies |e^{\lambda_i}| < 1$$

Thus, having a purely imaginary eigenvalue in the continuous case exactly corresponds to having an eigenvalue on the unit circle in the discrete-time map g^1.
Moving on, notice that this change of coordinates does not allow us to picture the trajectories, because the eigenvalues λ_1, λ_2 may be complex numbers (which do not belong to the sphere). Hence, we want to make a reasonable change of coordinates, that is, we want to still be in a matrix with real coefficients. In this case, we could change our system to the following forms of A:

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \begin{pmatrix} \mu & \sigma \\ \sigma & \mu \end{pmatrix}$$

after the change of coordinates and where $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\mu, \sigma \in \mathbb{R}$. Consider the equation

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \dots \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

where the matrix has the form of <u>one of the two matrices</u> above (though, the Jordan block $\left(\begin{smallmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{smallmatrix}\right)$ is also possible). There are a few cases for hyperbolic maps. A visualization of the dynamics for these cases is given previously.

1. If we have $\left(\begin{smallmatrix} \mu & \sigma \\ \sigma & \mu \end{smallmatrix}\right)$ and $\mu = 0$, then the dynamics around the point will look like concentric circles. In this case, we do not have a hyperbolic system, resulting from the fact that our eigenvalues are purely imaginary. We have a **center**.

2. If we have $\left(\begin{smallmatrix} \mu & \sigma \\ \sigma & \mu \end{smallmatrix}\right)$ and $\mu \neq 0$, then we have a spiral shape. If $\mu < 0$, then we are going towards the point as time increases, and vice versa for $\mu > 0$. In either case, the system is hyperbolic, and the point is called a **focus**.

3. If we have $\left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)$ and $\lambda_1 < 0 < \lambda_2$, then we will have a **saddle**. In this case, the dynamics around our point will actually represent hyperbolas, and this case is the reason for the name, hyperbolic point. Further, the four trajectories along the axes (going directly in and out along horizontal and vertical directions) are called *separatrices*.

4. If we have $\left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)$ and $\lambda_1 \cdot \lambda_2 > 0$, then we have a **node**. This is a similar case to that of the focus. We can have an attracting or repelling node depending on the signs of λ_i (repelling for positive, attracting for negative). This case is also hyperbolic.

5. If we have $\left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)$ and $\lambda_1 \cdot \lambda_2 = 0$, then we have a **degenerate point** and this case is not hyperbolic. There are many possible trajectories in this case. We usually omit consideration of this case in linear systems and reserve its investigation for nonlinear systems.

15.3 Grobman-Hartman Theorem

We state the theorem.

Theorem. *Grobman-Hartman Theorem*
Hyperbolic fixed points have a structurally stable neighborhood.

How do we understand this theorem? Suppose we have a vector field ν_1 with a hyperbolic point (say, a saddle). We may slightly perturb our map so that we have a vector field ν_2. Let U_1, U_2 be respective neighborhoods around each of the hyperbolic points in these vector fields. There exists a homeomorphism $h : U_1 \to U_2$, which maps trajectories of first vector field to the second vector field. Hence, in ν_2, we will have a similar picture of dynamics.
Based on this theorem, we may prescribe a process of investigation, given some vector fields.

1. Find fixed points.

2. Find linearization.

So, we begin to investigate each fixed point independently. Change coordinates so that the fixed point is at the origin, then we will have

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + O(x^2 + y^2)$$

where $O(x^2 + y^2)$ is some nonlinear part. If the matrix A is hyperbolic, then we may ignore this nonlinear term. In the case where A is not hyperbolic, we may have unpredictable results. For instance, a center point may no longer be a center after a small change (i.e. maybe it turns into a spiral).

Example. Consider the following system

$$\begin{cases} \dot{x} = x^2 \\ \dot{y} = y \end{cases}$$

We have two eigenvalues, 0 and 1. This is an example of the so-called *saddle node*. The naming is because on one side, the dynamics look like a saddle and on the other side, they resemble that of a node. What happens if we have $\dot{x} = x^3$ instead?

15.4 On Periodic Trajectories

Suppose we have a periodic trajectory on a plane. Then, this trajectory is closed. We employ a Poincaré map to investigate this trajectory. Pick a transversal line at some point on this trajectory, this line should have no tangencies to the vector field. We consider the first return map: starting from a point on the transversal, we consider the shape of the trajectory as it returns to the transversal. We will call the transversal line Γ, and denote by $P : \Gamma \to \Gamma$ the Poincaré map. Note that P is not a flow or a vector field, it is a diffeomorphism. Choose a point 0 on the transversal so that $P(0) = 0$ (such a point exists because we began with a cycle, a periodic trajectory). This will be a fixed point.

Suppose we are in the one-dimensional case. If $P'(0) \neq 1$, then we have a hyperbolic cyclic. Otherwise, it is not hyperbolic. If $P'(0) < 1$, then we have an attracting cycle and all points go towards the cycle. If $P'(0) > 1$, then we have a repelling cycle and all points (in a neighborhood) go away from the cycle. As a consequence of this behavior, there are no other cycles, and for this reason we call these **isolated cycles** or **limit cycles**. Suppose that we are in the non-hyperbolic case. It is possible for limit cycles to exist. The simplest case is the parabolic cycle where

$$P(x) = x + ax^2 + \dots, \;\; a \neq 0, \, x^2 < 1$$

There are other complicated cases, for instance, we may have infinitely many cycles that are each tending one to another. Still, between these, we have winding and noncycles. There may even be some open area in the phase space which consists only of cycles (which cannot be isolated). We may have a situation where we have a fixed points (saddles) and two other fixed points which are centers.

Such patterns are common when we have what is called a **Hamiltonian system**. A Hamiltonian system is given by a function $H(x, y)$ satisfying

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial y} \\ \dot{y} = -\frac{\partial H}{\partial x} \end{cases}$$

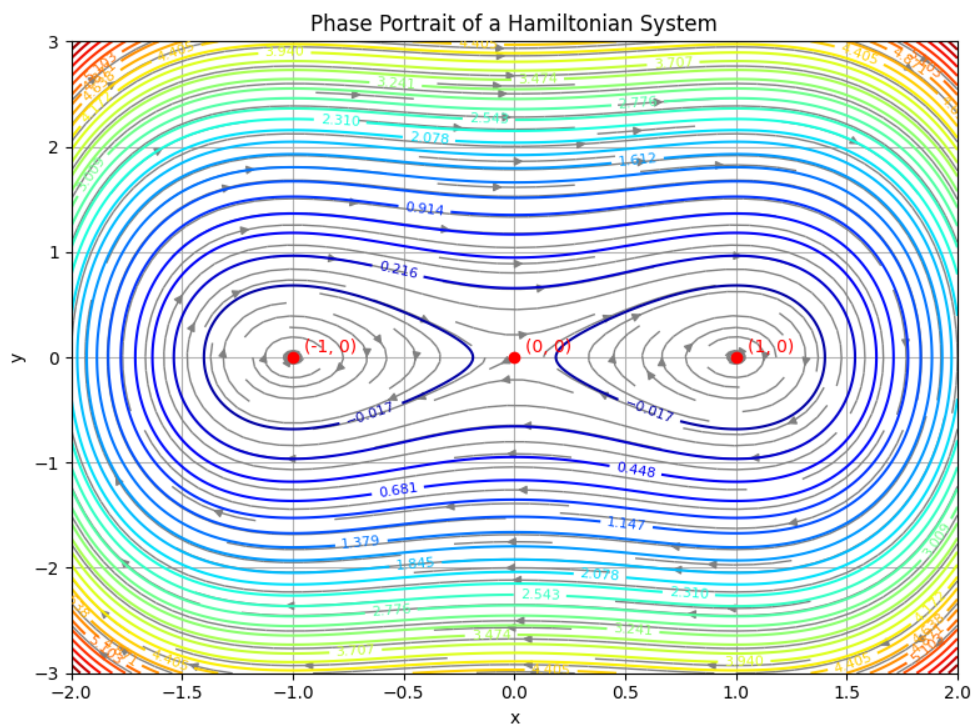
By the chain rule

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} = 0$$

Hence, H is constant for solutions to this equation. In many Hamiltonian systems, the level sets are closed curves, leading to periodic trajectories. This is why we might see families of cycles in the phase portraits of Hamiltonian systems, in contrast to isolated limit cycles in dissipative systems. As an explicit, example, let us consider

$$H(x, y) = \frac{1}{2}y^2 + \frac{1}{4}x^4 - \frac{1}{2}x^2$$

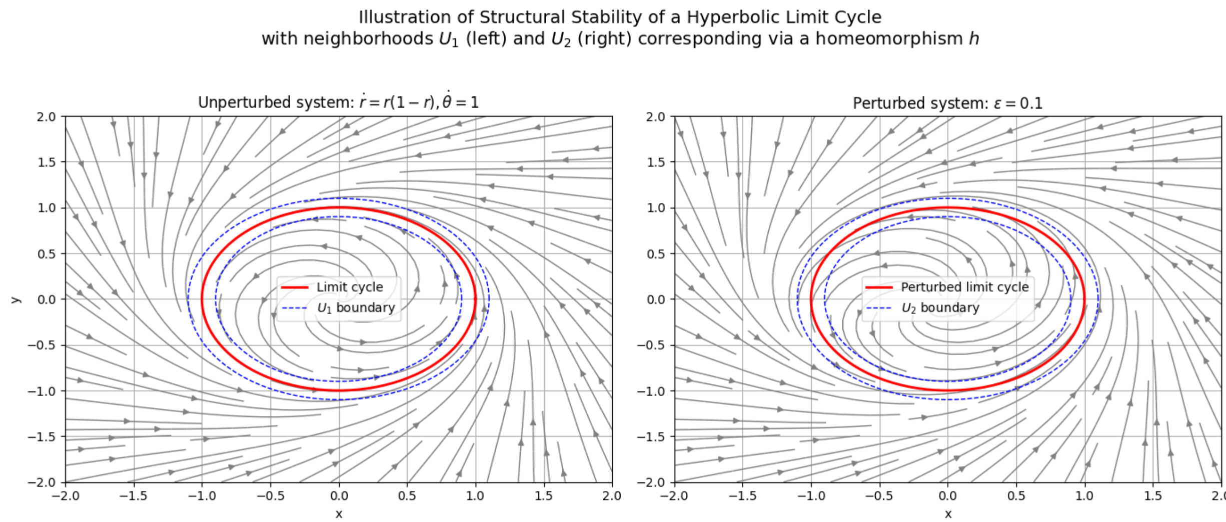
which has the following plot:



Above, we have discussed limit cycles in order to move to a more general situation. In the generic case, we have an analogue of the statement of the Grobman-Hartman theorem.

Proposition. Hyperbolic limit cycles are structurally stable.

Remark. Consider a vector field ν_1 with a nearby vector field ν_2 (i.e. vector field under a small perturbation). The above statement means that in a small neighborhood U_1 of the limit cycle in ν_1, there exists a corresponding neighborhood U_2 in ν_2 and a homeomorphism $h : U_1 \to U_2$ which maps the limit cycle in U_1 into one in U_2.

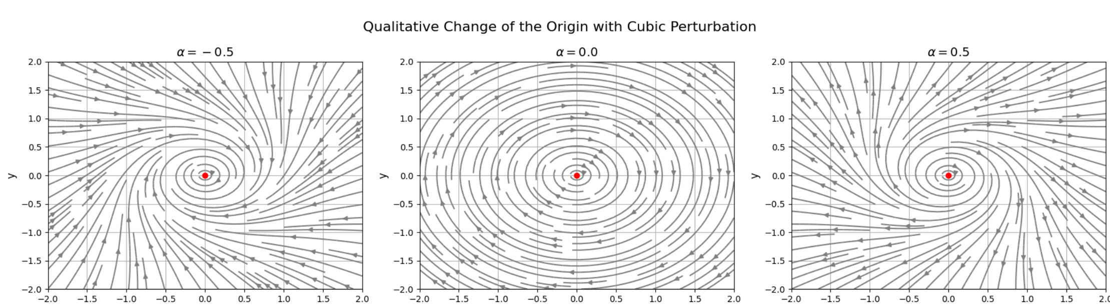


Remark. Why are hyperbolic limit cycles so important? The direct answer is because they are isolated and structurally stable. Notice that in the general case, finding a cycle is a very complicated issue. For instance, even for quadratic vector fields, it is unknown how many cycles we might have. Given

$$\begin{cases} \dot{x} = P(x,y) \\ \dot{y} = Q(x,y) \end{cases}$$

and $\deg P = 2$ and $\deg Q = 2$, we do not know how many cycles there will be, and we know that there are at least 4, some examples have 5.

Another interesting phenomenon is that of the **Andronov-Hopf bifurcation**. Initially, we might have a hyperbolic point such as a center and upon the addition of a small quadratic or cubic term, the dynamics are completely altered. For instance, we might have new foci.



15.5 Relation to Poincaré-Bendixson Theorem

Let us recall the Poincaré-Bendixson theorem. Suppose we have a vector field with a finite number of singular points. In this case, the ω-limit set of any point can only be

1. A singular point,
2. a cycle,
3. a polycycle.

Suppose we have a case where every singular point is hyperbolic. Then, denoting these singular points $s_1, s_2, \ldots, s_n$, we will have a *hyperbolic polycycle* which is connected by separatrices between the points. When a trajectory goes from one saddle to another, we have what is called a **saddle connection**. We forbid this situation for Andronov-Pontryagin. In the Andronov-Pontryagin conditions, observe that we have excluded the possibility of polycycles to eliminate saddle connections. Thus, all ω-limit sets must be hyperbolic points or hyperbolic cycles.

15.6 Proof Sketch of Andronov-Pontryagin

Lemma. A set of vector fields satisfying the AP conditions is open and dense .

Proof. Call H the set of vector fields on $\mathbb{S}^2$ such that all fixed points are hyperbolic. We claim that H is open and dense. To proceed, we need to recall Sard's lemma.

Aside. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a $\mathscr{C}^1$ map. If the rank of Df at x is less than m, then we call x a *critical point* and $f(x)$ the *critical value*. For instance, if $f : \mathbb{R}^2 \to \mathbb{R}$ is the constant function, then every point is a critical point, and the constant value is the critical value.

Sard's lemma. The Lebesgue measure of the set of all *critical values* is zero.

Returning from our aside, first, note that our vector field may be thought of as a map from the sphere $v : \mathbb{S}^2 \to \mathbb{R}^2$, given by

$$(x_1, x_2) \mapsto \begin{pmatrix} v_1(x_1, x_2) \\ v_2(x_1, x_2) \end{pmatrix}$$

Suppose that (x_1, x_2) is a singular point. Since it is a fixed point, it is mapped by the vector field to $(0, 0)$. Now, if $J_x v = 0$, then the point x is degenerate. If a point x is degenerate, then x is a critical point of the function v. By Sard's lemma, there exists a vector $\varepsilon \in \mathbb{R}^2$

such that 0 is a non-critical value for $v - \varepsilon$ (and we may choose ε to be arbitrarily small). To see why this is true, we can argue by contradiction and see that we end up for a set of critical values which has nonzero Lebesgue measure. (To be explicit, suppose that for all ε in some neighborhood, ε is a critical value...) As a result, it is possible to choose ε so that any singular point x is non-degenerate for $v = \varepsilon$. After a small perturbation, we obtain a vector field such that all zeroes are non-degenerate.

Recall in our classification, the center is the only non-degenerate and non-hyperbolic point. Our next goal should be to exclude centers. The general form of a center will be

$$v(x) = Ax + R(x)$$

where A is a matrix with purely imaginary eigenvalues $\pm i\omega$, $R(x)$ represent higher order nonlinear terms (think Taylor series), and x is a singular point. From the Taylor theorem, there exists a neighborhood U such that $|R(x)| < \frac{1}{2}|Ax|$. Let us specify another neighborhood V which contains x and such that $V \subseteq U$. We define a function φ which satisfies $\varphi(V) = 1$, $\varphi(U \backslash V) \in (0, 1)$ and $\varphi = 0$ for all values outside of U. We would like to perturb our map by setting $v_\delta = v + \delta x \varphi$. We claim there is only one singular point. For $x \neq 0$, we have

$$|v_\delta(x)| \geqslant |Ax| - |R(x)| - |\delta x \varphi| \geqslant \frac{1}{2}|Ax| - \delta|x| > 0$$

Hence, the only singular point is 0. We consider the linear part of v_δ at 0, which is

$$J_0 v_\delta = A + \delta I$$

Now, if $\delta > 0$, we have a stable focus, and if $\delta < 0$ we have an unstable focus. In this way, we have changed our center into a focus. To summarize our procedure above, we begin with a vector field v.

1. We perturb v by ε to eliminate all degenerate points.

2. We perturb this new field by δ to eliminate all centers. Here, we have that all singular points are hyperbolic. Note that all hyperbolic points are isolated. Further, the set of isolated points of a sphere is a finite set. Hence, this field must have only a finite number of hyperbolic points.

3. It is possible to perturb this map which has a finite number of hyperbolic points so that these points remain hyperbolic under that perturbation. When we are far away from hyperbolic points, there are no singular points. From Grobman-Hartman, we may find neighborhoods around our hyperbolic points which are structurally stable, and this allows us to find an open set of vector fields.

4. For any vector field, we may find an arbitrarily close vector field with only hyperbolic points. This shows that H is dense.

At this point, we have shown the Andronov-Pontryagin conditions in the case where we have hyperbolic fixed points. We will cover the cases of limit cycles and where there are no saddle connections later.

On Structural Stability. A structurally stable vector field belongs to the set H (the set of vector fields where all fixed points are hyperbolic). To see why this is true we consider the contrapositive, suppose a vector field does not belong to H. Then this vector field must either have degenerate points or centers. If the vector field has centers, then under perturbation we may obtain a vector field with a focus. These vector fields cannot be topologically conjugated since the former has closed trajectories accumulating to the fixed point, while no such trajectories exist in the latter vector field. So there is no homeomorphism and no structural stability. Moreover, it turns out that under perturbation, it is possible for a vector field with degenerate points to give rise to two fixed points, meaning that the vector field is not structurally stable.

15.7 Exercises

1. Describe a map g^1 for center and a linear system.

2. Trajectories of a focus and of a node are homeomorphic.

16 April 9, 2025

16.1 Continuation: Andronov-Pontryagin Criteria

Recall the conditions below for a vector field:

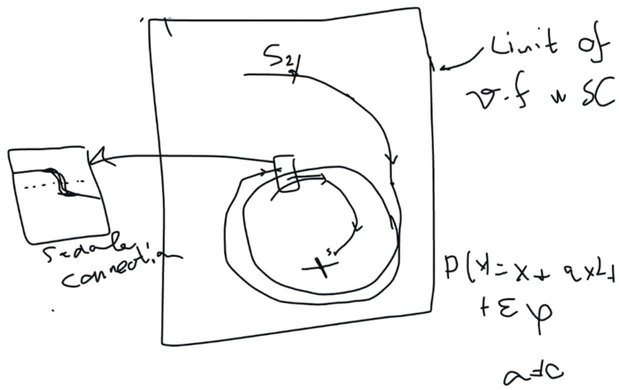
1. All fixed points are hyperbolic.
2. There are no saddle connections.
3. All periodic trajectories are limit cycles and hyperbolic.

These three properties define Andronov-Pontryagin vector fields and on the sphere they describe an open and dense set in the space of all vector fields with respect to $\mathscr{C}^1$ topology.
Remark. If we exclude saddle connections, we also exclude all polycycles (if there is a polycycle, then there must be a connection between two saddles, as only saddles have trajectories which go towards and away). We also forbid saddle loops (saddle connections of a point to itself) as we consider these to be saddle connections.

We said in our previous class that the set of vector fields that satisfy the three conditions above are open and dense, and they are structurally stable. Let H be the set of vector fields such that all singular points are hyperbolic. Previously, we showed that H is open and dense, we also saw that membership in H is a necessary condition for structural stability. (It turns out structural stability $\iff$ three conditions of AP).

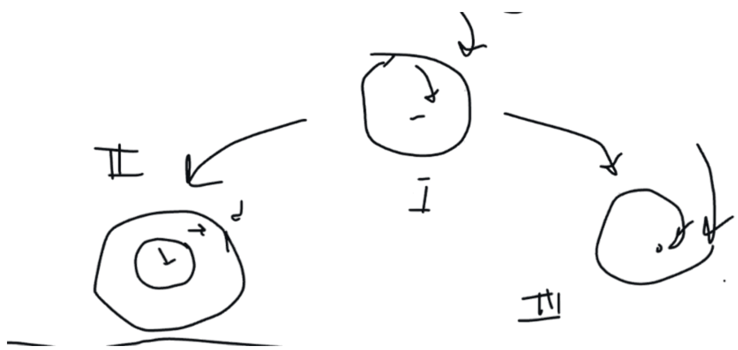
Moreover, if we take the space of vector fields where all limit cycles are hyperbolic, this set is not open. In the same way, if we take the set of the vector fields with no saddle connections, this set is also not open. We illustrate these cases.

1. We begin with a parabolic cycle. A parabolic cycle is *not* hyperbolic. It is a cycle where some trajectories go out from the cycle and other trajectories go toward the cycle. The derivative of the Poincaré map $P(x) = x + axL$ is equal to 1.



Suppose that for each of these trajectories going away from and toward the cycle, they are separatrices of a saddle. If the set of vector fields with no saddle connections is open, then the complement of the set should be closed. Thus, for any sequence of fields with saddle connections, the limit should also contain saddle connections. We will show that the above image is the limit of vector fields with saddle connections.

How can we do this? We may perturb our map $P(x) + x + axL + \varepsilon\varphi$, where φ is a function near this parabolic cycle. As a result, this parabolic will disappear. When we perturb a parabolic cycle, there are three possible outcomes: (1) the cycle persists, (2) the cycle splits into two hyperbolic cycles, or (3) the cycle disappears entirely.

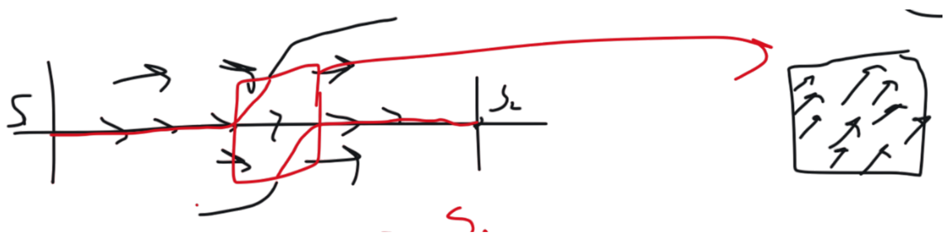


The key idea is that it is possible under perturbations, since our trajectories are very close to each other, to merge the trajectories so that they become one trajectory, and so that this trajectory connects the two saddles. So, since our original picture with a parabolic cycle is not in the set of vector fields with saddle connections, this set is not closed. Hence the set of vector fields with saddle connections is not open.

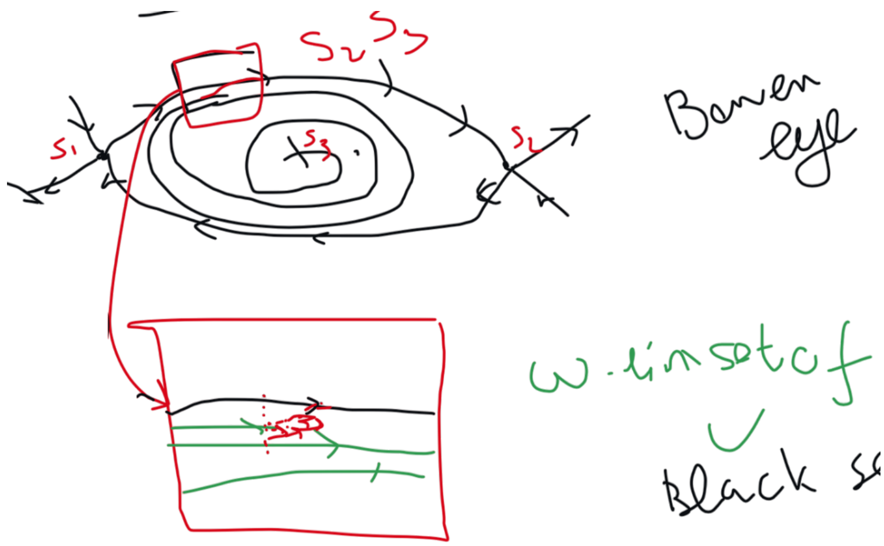
2. The set of vector fields where all limit cycles are hyperbolic can also be shown to not be open, but it requires more calculation and we omit the explanation.

These examples illustrate that these two conditions do not define an open set alone, but together they may define an open set. We now discuss why these sets are dense.

1. Take the space of vector fields with no saddle connections. We restrict further to the case where there are a finite number of fixed points which are all hyperbolic, call a set with these properties H. Then, H has a finite number of saddles, say n saddles. Each saddle can have at most four separatrices, so the maximum number of saddle connections is $2n$ (two separatrices per saddle connection), and this number is also finite. Consider the situation of a separatrix between two saddles below.



It is possible to choose an arbitrary point, in the red region, and to change our field by a small perturbation (it may be done smoothly, see the procedure we used to remove centers). How can we guarantee that when breaking these saddle connections, no new saddle connections are formed? For instance, we may have this situation when we have Bowen's eye and a saddle in the center.



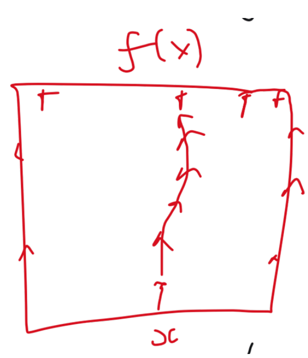
In the image, breaking the saddle connection between s_1 and s_2 may lead to the formation of a new saddle connection between s_3 and s_2. Surprisingly, this newly created saddle connection may actually be destroyed without the creation of another saddle connection. The second saddle connection leads to Bendixson's trap (i.e. the contradictory situation Poincaré-Bendixson's theorem). The new saddle connection cannot be the ω-limit set of any point, meaning that the new separatrix is isolated from all other separatrices.

Thus, through small perturbations in our vector field, we are able to reach a vector field with no saddle connections. So the set of vector fields with no saddle connections is a dense set. Also, requiring no saddle connections is necessary for structural stability. If the field has non-hyperbolic points, we are done because non-hyperbolic points destroy structural stability. If all points are hyperbolic, then in a small neighborhood of this vector field, there is another vector field with no saddle connections. We remark that saddles and trajectories between them are invariant under topology conjugacy. Since we can find a vector field with no saddle connections close to a vector field without them (density), vector fields with saddle connections cannot be structurally stable.

Q: Using the same density argument, why is the set of fields with saddle connections not dense in the same way as the set of fields without saddle connections?

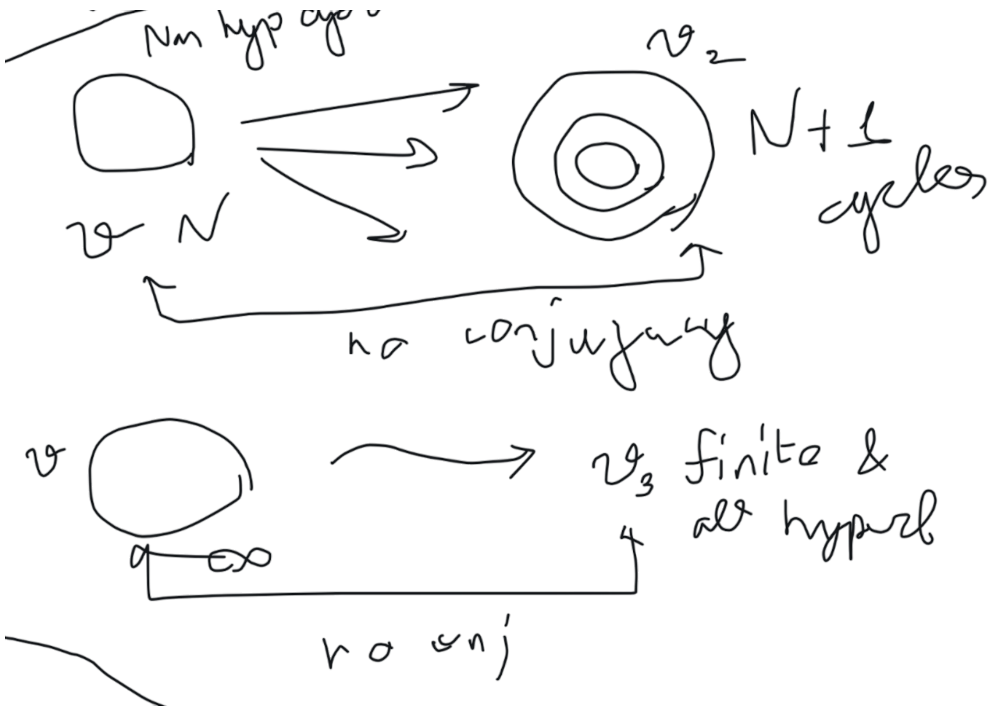
2. We discuss the case of limit cycles now. Start with a limit cycle, a Poincaré map P, and two trajectories, one on the outside of the limit cycle and another on the inside. Create an annulus between these two trajectories that includes the limit cycle. We want to, through a perturbation, make the limit cycle into hyperbolic cycles or to make it disappear entirely.

Denote the Poincaré map $P(x) = x + \ldots$. We will change it to $P(x) = \varepsilon + x + \ldots$ and aim to change the limit cycle into two hyperbolic cycles or no cycle at all. This ε exists due to Sard's lemma. However, we cannot arbitrary change our Poincaré map. We must change our vector field to achieve a similar result. To start, draw a rectangle around the limit cycle (indicated in red). It may be done smoothly so that the field is the same on the boundary.



Changing the field in this rectangle allows us to change the field in our annulus. After this, we obtain our desired Poincaré map where we have either no hyperbolic cycles or hyperbolic cycles. Argument slightly unclear, revisit at 43:00. It is important to have finiteness on the number of limit cycles, because if there are infinitely many limit cycles, then making ε-small perturbations may lead to an uncontrolled change overall. All perturbations should be done simultaneously. We have a set of periodic nontrivial trajectories which is compact. Why? If we have a center, then we have infinitely many periodic trajectories accumulating to the center. However, since we have all hyperbolic limit cycles, this is not the case. Another possibility is that periodic points go to a polycycle (such as in the accumulation of

saddle loops in the Hamiltonian picture). It can be demonstrated that if we have no saddle connections and no centers, then the set of periodic trajectories is a compact set. This allows us to have a finite number of annuli to cover our trajectories.
After this perturbation, all cycles are hyperbolic and isolated, meaning that there are only a finite number of them. To see that hyperbolic limit cycles are a necessary condition for structural stability, we assume we have non-hyperbolic cycles and show there is no structural stability.



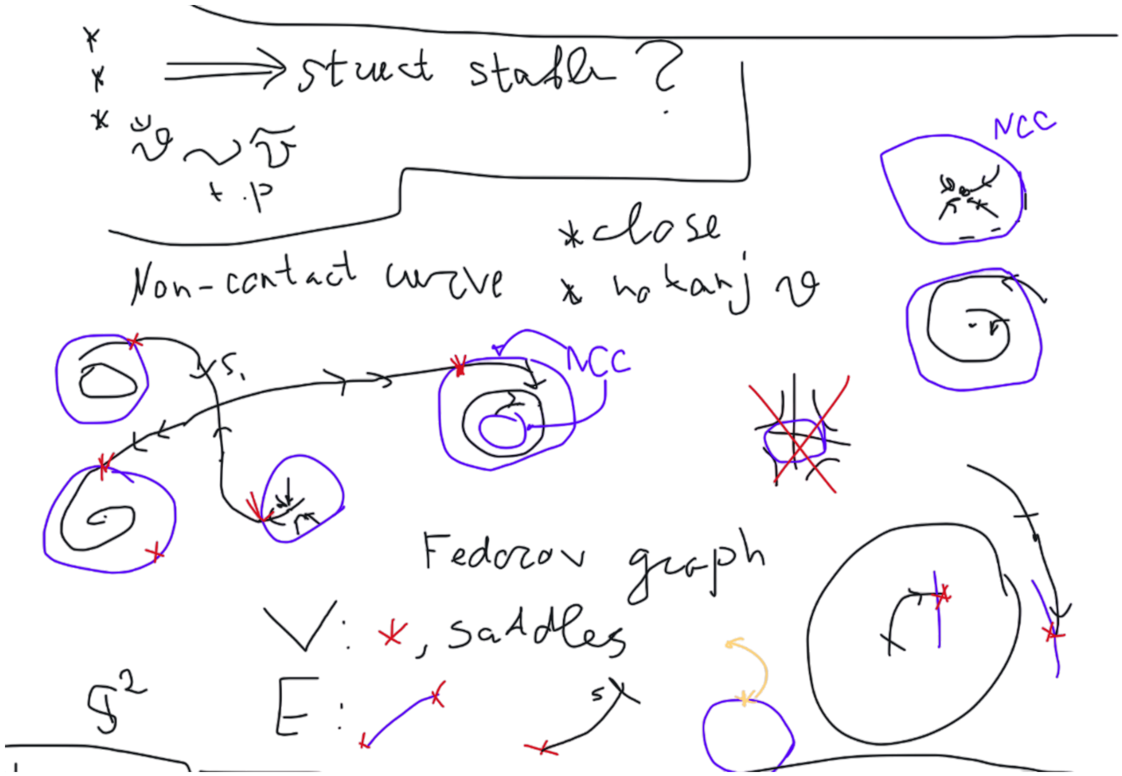
16.2 Sketch of Sufficiency of Andronov-Pontryagin

Why are these three conditions sufficient for structural stability. Note that if a field in the set of vector field satisfying these three conditions is structurally stable, then this implies that it is open. Conjugacy will preserve these three properties. So after showing sufficiency for structural stability we will be done.

To proceed, our strategy is to work with a characteristic graph, and we define a non-contact curve (NCC).

Definition. *Non-contact curve*
A closed curve which has no tangencies to a vector field v.

An example of a NCC might be a seen in a circle around a node or a focus.



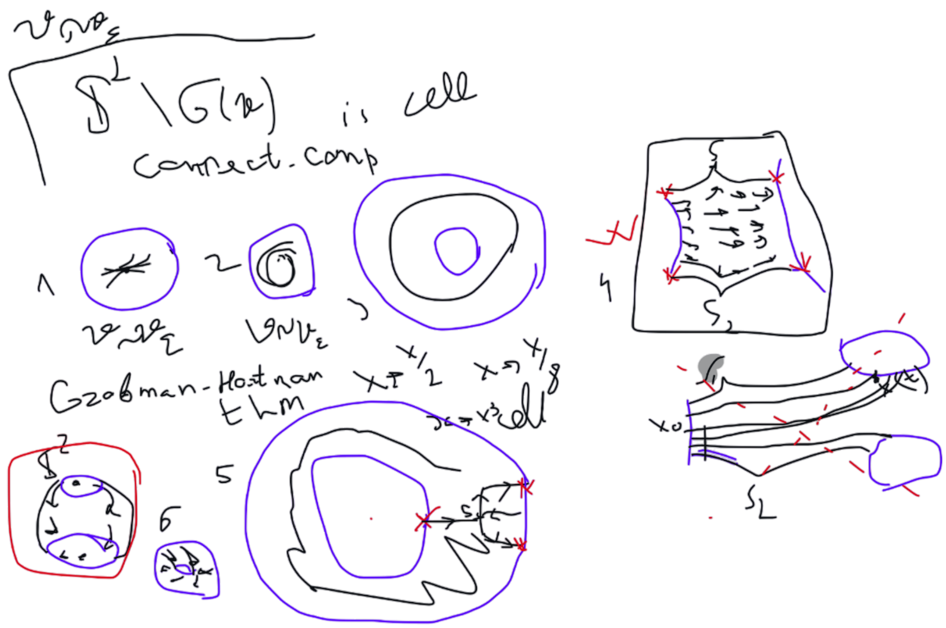
If we have a hyperbolic cycle, for example an attracting hyperbolic cycle, we can create two NCC around this cycle. It is impossible to make a NCC around a saddle. Since our condition requires that there are no saddle connections, this means that all separatrices from a saddle are connected to some other hyperbolic point, either a focus or a node. We draw NCC around all of these other hyperbolic points and exclude our area of consideration to the regions outside of what is bounded by these NCCs.

We define a graph which is called the characteristic graph or Fedorov graph. The *vertices* of the graph are the points of intersection between the NCC and separatrices (indicated by red stars in the image) and the saddles themselves. The *edges* of the graph are the lines between two red points as well as the separatrices. This graph is just a planar graph on a sphere.

Lemma. Consider the Fedorov graphs $G(v)$ and $G(v_\varepsilon)$ of a vector field and its perturbation. These graphs are the same abstract graph. If the set of all non-contact curves are the same, then the graphs $G(v)$ and $G(v_\varepsilon)$ are close.

Remark. The graphs $G(v)$ and $G(v_\varepsilon)$ are homeomorphic (saddles are preserved, separatrices are preserved, NCC preserved, and so forth). All vertices and edge labels are preserved under the homeomorphism.

Our goal is to prove that the vector fields v and v_ε, given that that are close enough, may be conjugated. We start with a homeomorphism between graphs and we would like to extend our result to be a homeomorphism on the entire sphere. We extend this homeomorphism by parts. Consider the set $\mathbb{S}^2 \backslash G(v)$. We will call any connected component of this set a cell. There are a few situations that may arise for these cells, pictured below.

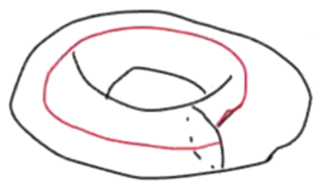


1. We have a cell that is a disk containing a focus or a node. In this case, shown in (1) and (2), our vector fields v and v_ε are conjugated as the perturbation is small to remain within the neighborhood in which these points are structurally stable.
2. Another option is a hyperbolic cycle resides in the cell, either attracting or repelling. We know that hyperbolic limit cycles are structurally stable. See the Grobman-Hartman theorem.
3. The fourth situation is labeled, where we are bounded between edges, some which are parts of NCCs and others which are separatrices connecting to saddles. This is the only possible picture. If these trajectories go from one NCC to two different NCCs, there is a problem of continuity of solutions. The point of intersection must change continuously and it cannot 'jump' from one point to another. It may be demonstrated, through a theorem about the *rectifiability of vector fields*, that all regions of this type may be conjugated between themselves.
4. Our saddle may between two NCCs.
5. We have one attracting and one repelling point on the entire sphere. The region in between these points has no singular points or cycles, just a nonzero vector field.

17 April 23, 2025

17.1 Diffeomorphisms on the Circle and Torus

We return to diffeomorphisms and flows on the circle and torus. We have discussed previously that flows on the torus with no singularities are connected with Poincaré maps.



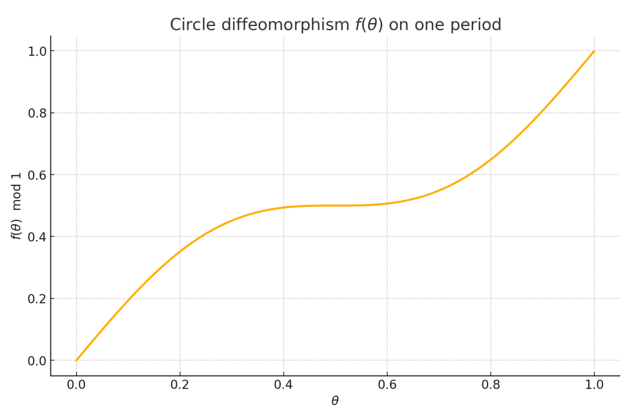
In the above picture, we have selected a nontrivial circle (cross-section) from the torus. We notice that the first return map on the torus maps one point on the circle to another point. We may call this map $f_1 : S^1 \to S^1$. We will be primarily interested in these circle diffeomorphisms. Also, we will restrict our consideration to orientation preserving maps only.

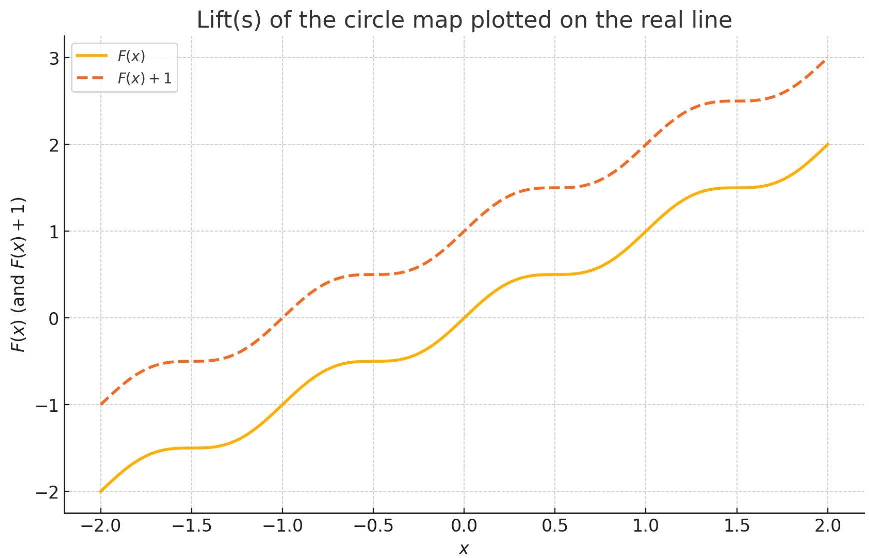
17.2 Lift and Rotation Number

Definition. *Lift*
Let $f : S^1 \to S^1$ be a circle diffeomorphism. We say that f lifts to F if, given the natural projection $\pi : \mathbb{R} \to S^1$ by $x \mapsto \{x\}$, we have the following equation

$$f \circ \pi = \pi \circ F$$

It is possible to create a lifting F by taking the map $f : [0, 1) \to [0, 1)$ and translating with $F(x) = f(\{x\}) + [x]$. Note that F is not unique, as $F + n$, where n is some integer, will still be a lifting. However, all liftings belong to this family of maps. Hence, the difference between any two liftings is a constant integer map.





As an example, consider the circle diffeomorphism

$$f(\theta) = \theta + \frac{\sin(2\pi\theta)}{2\pi} \pmod 1, \quad \theta \in [0, 1)$$

This is the first picture. The lift, given by $F(x) = x + \frac{\sin(2\pi x)}{2\pi}$ is shown in the second picture, along with another valid lift represented by the dotted line.

Definition. *Rotation Number*
The rotation number is denoted by τ and takes in F as an argument. We write

$$\tau(F) = \lim_{n\to\infty} \frac{F^n(x) - x}{n}$$

Remark. What is the intuitive idea behind this formula? We may think physically. The point of the rotation number is to represent the average drift or rotation of a map. Note that the rotation number may differ depending on the lifting which is chosen, but this difference will be a constant. So, the rotation number will be a family of numbers, but they will all have the same fractional part.

Proposition. Let $f : S^1 \to S^1$ be a circle diffeomorphism.

1. The rotation number $\tau(f)$ does not depend on x.
2. $\tau(f)$ exists.

Proof. Consider distinct points $x, y \in \mathbb{R}$. Observe that

$$\frac{F^n(x) - x}{n} - \frac{F^n(y) - y}{n} = \frac{F^n(x) - F^n(y)}{n} + \frac{y - x}{n}$$

Since f is a diffeomorphism, if $x, y \in [0,1]$, then $F(x) - F(y) \in [0,1]$. Inductively, we may conclude that $F^n(x) - F^n(y) \in [0,1]$. It follows from taking the modulus that

$$\left|\frac{F^n(x)-x}{n} - \frac{F^n(y)-y}{n}\right| \leqslant \left|\frac{F^n(x)-F^n(y)}{n}\right| + \left|\frac{y-x}{n}\right| \leqslant \frac{2}{n} \xrightarrow[n\to\infty]{} 0$$

Thus, the limit does not depend on the points.
To show existence, we intend to show that the sequence $\frac{F^n(x)-x}{n}$ is Cauchy. Take $x \in [0,1)$. We denote the terms of the sequence $\gamma_n = \frac{F^n(x)-x}{n}$. By hypothesis, we have

$$F^{m+n}(x) - x \leqslant 1 + F^m(x) - x + F^n(x) - x$$

By the division algorithm, we may write any integer l as $l = nk + r$. Applying the elementary bound above recursively k times, we have:

$$\begin{aligned}
F^{nk+r}(x) - x &\leqslant 1 + \left[F^{n(k-1)+r}(x) - x\right] + \left[F^n(x) - x\right] \\
&\leqslant 2 + \left[F^{n(k-2)+r}(x) - x\right] + 2\left[F^n(x) - x\right] \\
&\ \vdots \\
&\leqslant k + \left[F^r(x) - x\right] + k\left[F^n(x) - x\right].
\end{aligned}$$

Now, substituting in, we have

$$\begin{aligned}
\frac{F^l(x)-x}{l} &\leqslant \frac{k(F^n(x)-x)}{l} + \frac{F^r(x)-x}{l} + \frac{k}{l} \\
&\leqslant \frac{F^n(x)-x}{n} + \frac{F^r(x)-x}{nk} + \frac{1}{n} \\
&\leqslant \frac{F^n(x)-x}{n} + \frac{2}{n}
\end{aligned}$$

where the second inequality in the chain follows from $\frac{1}{nk+r} < \frac{1}{nk}$ and the third inequality from $F^r(x) - x \leqslant (x+k) - x$. Since we may choose n, for any $\varepsilon > 0$, we see that $\gamma_l - \gamma_n \leqslant \frac{2}{n} < \varepsilon$. We have shown so far that $\lim \gamma_n \in \mathbb{R} \cup \{-\infty\}$. We need to exclude the case of $-\infty$. Consider that

$$\begin{aligned}
\frac{F^n(x)-x}{n} &= \sum_{i=1}^{n} \frac{F^i(x) - F^{i-1}(x)}{n} \\
&= \sum_{i=1}^{n} \frac{F^{i-1}(F(x)) - F^{i-1}(x)}{n} \geqslant \min_{0\leqslant y\leqslant 1} F(y) - y
\end{aligned}$$

Therefore, the limit is real.
At this point, we prove some of the relations we used earlier. Observe that

$$\begin{aligned}
F^{m+n}(x) - x &= F^{m+n}(x) - F^n(x) + F^n(x) - x \\
&= \underbrace{(F^m(F^n(x)) - F^m(x+k))}_{\in[0,1]} \\
&\quad + (F^m(x+k) - (x+k)) + \underbrace{(x+k-F^n(x))}_{\in[-1,0]} + F^n(x) - x
\end{aligned}$$

The above statement holds for any k, and we will choose $k = [F^n(x) - x]$, the integer part of this difference. Hence, $x + k \in [F^n(x) - 1, F^n(x)]$. Based on this, we see that $(x + k - F^n(x)) \in [-1, 0]$. Since $x + k, F^n(x) \in [k, k+1]$, we have seen before, via induction, that $(F^m(F^n(x)) - F^m(x+k)) \in [0, 1]$. Thus, we may continue the previous inequality

$$\begin{aligned} &\leqslant 1 + F^n(x) - x + F^m(x+k) - (x+k) \\ &\leqslant 1 + F^n(x) - x + F^m(x) - x \end{aligned}$$

where the last inequality follows from the fact that $F - \mathrm{Id}$ is periodic. Need to check why last inequality holds more closely. This proves the inequality we used earlier in our proof of the proposition. ■

17.3 Applications of Rotation Number

Lemma. $\tau(f) = 0$ if and only if f has a fixed point.

Proof. Suppose that f has a fixed point, so $f(a) = a$. It is possible to choose a lifting so that $F(a) = a$. We have

$$\frac{F^n(a) - a}{n} = 0$$

for all $n \in \mathbb{N}$. Since the rotation number does not depend on an individual point, we see that $\tau(f) = 0$. On the other hand, suppose that $f : S^1 \to S^1$ and $\tau(f) = 0$. Assume, for contradiction, that

$$F(x) - x \neq 0$$

for all x. Without loss of generality, assume that $F(x) - x > 0$. Since $F(x) - x$ is periodic, it will have a minimal value on $[0, 1]$, so $F(x) - x > \delta$ for some real number $\delta > 0$. Recall

$$\frac{F^n(x) - x}{n} = \sum_{i=1}^{n} \frac{F^i(x) - F^{i-1}(x)}{n} > \frac{n\delta}{n} = \delta$$

Hence, $\lim_{n\to\infty} \frac{F^n(x)-x}{n} \geqslant \delta$. But, we assumed that $\tau(f) = 0$ ($\lightning$). ■

Remark. In an informal and more intuitive manner, suppose we have a map on a circle. Suppose also that we state that the drift of any point is always in the same direction (i.e. clockwise), and it always drifts by at least δ. Therefore, the average drift would be at least δ as well. This is essentially what the lemma is saying.

17.3.1 On Rational and Irrational Rotation Numbers

Rotation numbers τ may be either rational or irrational. In the case where τ is rational, we always have a periodic orbit. Further, all periodic orbits will have the same period, which coincides with the denominator of the rational number. As an example, we may consider the

North-South map and call it g. Since this map has fixed points, by the Lemma above, we know that its rotation number is 0. Consider the map $x \mapsto g(x) + \frac{1}{2}$. For this map, we have an orbit $\{S, N\}$, and computing the rotation number will show that $\tau = \frac{1}{2}$. We see that the period of orbit matches the denominator of τ.

By contrast, suppose τ is irrational. What is important in this case is the smoothness of the map. If $f \in \mathscr{C}^2$, then f is topologically conjugated to an irrational rotation. It turns out it will be conjugated to $x \mapsto x + \tau(f)$. This follows from **Denjoy's Theorem**. If $f \in \mathscr{C}^1 \backslash \mathscr{C}^2$, then there is the Denjoy counterexample.

Remark. In dynamical systems, the difference between $\mathscr{C}^1$ and $\mathscr{C}^2$ is usually very drastic (and connected to distortion control). This is an important topic with relations to Diophantine approximation and Liouville numbers.

Suppose that $\tau(f) = \frac{p}{q}$. What is $\tau(f^q)$? Let F be a canonical lifting of f, so that F^q is a canonical lifting of f^q. Observe that

$$\lim_{n\to\infty} \frac{F^{qn}(x) - x}{n} = q \lim_{n\to\infty} \frac{F^{qn} - x}{qn} = q \cdot \frac{p}{q} = p$$

Because rotation numbers are defined only up to integer numbers (i.e. p and 0 are in the same equivalence class), we see that $\tau(f^q) = 0$. Hence f^q has a fixed point. This implies that f has a periodic orbit of period q. The existence of a periodic orbit is useful for maps on a circle. To summarize, we have concluded that if we have a rational rotation number, then we have a periodic orbit of period q. On the other hand, if we have a periodic orbit, we also have a rational rotation number. Suppose that $f^q(a) = a$, observe that

$$\lim_{n\to\infty} \frac{F^n(a) - a}{n} = \lim_{n\to\infty} \frac{F^{mq}(a) - a}{mq}$$

If we have a periodic orbit, then we have a difference which is always an integer number (if we return to the same point on a circle, we return to the same point on the real line up to an integer). Say that $F^q(a) - a = p$ and $F^{2q}(a) - F^q(a) = p$. Now we see that

$$\lim_{m\to\infty} \frac{F^{mq}(a) - a}{mq} = \frac{mp}{mq} = \frac{p}{q}$$

So, we have shown the proposition below.

Proposition. A rotation number is rational if and only if there is a periodic orbit.

We continue our investigation of properties of the rotation number.

Proposition. The rotation number is a topological invariant.

Proof. Suppose we have two topologically conjugated maps f, g, so there exists a homeomorphism h, and

$$g = h^{-1} f h$$

When we say that the rotation number is topologically invariant, we mean that $\tau(f) = \tau(g)$. Let F and H be liftings of f and h, respectively. We may obtain a lifting H^{-1} of h^{-1} by considering

$$\pi H^{-1} = h^{-1}(h\pi)H^{-1} = h^{-1}(\pi H)H^{-1} = h^{-1}\pi$$

This shows that if h can be lifted to H, then h^{-1} may be lifted to H^{-1}. By taking a composition, we see that

$$g^n = h^{-1} f^n h$$

Let $x \in [0, 1]$. Then,

$$0 - 1 \leqslant H(x) - x < H(x) < H(1) = 2$$

We know that $H(x) - x$ is periodic. So, for all x with period 1, we have

$$|H(x) - x| < 2 \text{ and } |H^{-1}(x) - x| < 2$$

We will label $H(x)$ and $H^{-1}(x)$ both as y for the subsequent argument. We claim that if $|y - x| < 2$, then $|F^n(y) - F^n(x)| < 3$. To show this, we may reason that

$$\begin{aligned} -3 \leqslant [y] - ([x] + 1) = F^n([y]) - F^n([x] + 1) < F^n(y) - F^n(x) \\ < F^n([y] + 1) - F^n([x]) = [y] + 1 - [x] \leqslant 3 \end{aligned}$$

So, the inequality holds. Now, to show the equality of the rotation numbers, we want to consider

$$\frac{G^n(x) - x}{n} - \frac{F^n(x) - x}{n} = \frac{G^n(x) - F^n(x)}{n}$$

and show that this expression tends to 0 as $n \to \infty$. If is sufficient to show that $G^n - F^n$ is bounded. Observe that

$$|H^{-1}F^nH(x) - F^n(x)| \leqslant |H^{-1}F^nH(x) - F^nH(x)| + |F^nH(x) - F^n(x)|$$

Setting $y = H(x)$ and using the inequality we have proven before, we have $|F^nH(x) - F^n(x)| < 3$. Next, taking $a = F^nH(x)$, we have

$$|H^{-1}F^nH(x) - F^nH(x)| = |H^{-1}(a) - a| < 2$$

Altogether, we have

$$|H^{-1}F^nH(x) - F^n(x)| \leqslant 2 + 3 = 5$$

So $|G^n - F^n|$ is indeed bounded. Therefore, our rotation numbers are equal. ■

Remark. Note the contrapositive of this statement, that if two maps have two different rotation numbers, they cannot be topologically conjugated!

We now give an intuitive view of some facts.

1. We may think of the rotation number τ as a function on the set of diffeomorphisms on the circle. There is some sense of continuity (in a multifunctional sense) for τ.

2. Let f be a circle diffeomorphism and consider $f + \varepsilon$. If f has rational rotation number and is not strictly a rotation, then the rotation number of $f + \varepsilon$ does not change. On the contrary, if f has irrational rotation number, then the rotation number for $f + \varepsilon$ will be increasing or decreasing. In this case, it turns out the map $\tau(f)$ will resemble the *Cantor ladder* (a continuous function which is not smooth, not Lipschitz, but is Hölder).

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18.1 Continued Discussion on Rotation Number

In the previous lecture, we focused on rational rotation numbers. In this lecture, we will focus on irrational rotation numbers. We aim towards the Denjoy theorem. Previously, we used τ as the symbol for rotation, but we will replace this with ρ instead.

Theorem. *Denjoy's Theorem for Irrational Rotation*
Suppose f be an orientation-preserving $\mathscr{C}^2$ diffeomorphism of the circle and let $\rho(f)$ be its rotation number. $\rho(f) \in \mathbb{R}\backslash\mathbb{Q}$ if and only if f is topologically conjugated to $x \mapsto x + \rho$.

To begin, we need two preliminary lemmas.

Lemma about order. Let f be a map ($\mathscr{C}^1$, but this works for homeomorphisms) with a lifting F, and an irrational rotation number $\rho(F)$. For all integers n_1, n_2, m_1, m_2 and $x \in S^1$, the following two statements are equivalent:

1. $n_1\rho + m_1 < n_2\rho + m_2$
2. $F^{n_1}(x) + m_1 < F^{n_2}(x) + m_2$

Remark. In the formulation above, we see that there is some similar to conjugacy of f to irrational rotation.
Proof. Define the function

$$p(x) = F^{n_1}(x) - F^{n_2}(x) + m_1 - m_2$$

We claim that $p(x) \neq 0$. If not, then there exists some x such that $F^{n_1}(x) - F^{n_2}(x)$ is an integer. But then $f^{n_1}(x) - f^{n_2}(x) = 0$, which means that we have a periodic orbit, which is impossible. Hence, we must have $p(x) \neq 0$. Without loss of generality, suppose $p(x) < 0$ for all x. Equivalently, this means $F^{n_1}(x) + m_1 < F^{n_2}(x) + m_2$. Denote $y = F^{n_2}(x)$, by substitution we now have

$$F^{n_1 - n_2}(y) - y < m_2 - m_1$$

This may be shown to hold for all y, using the same argument by contradiction above, where we obtained a periodic orbit. Let $\alpha = n_1 - n_2$. We may write $F^{\alpha}(0) - 0 < m_2 - m_1$. Observe also that $F^{2\alpha}(0) < m_2 - m_1 + F^{\alpha}(0) < 2(m_2 - m_1)$. By induction, we have

$$F^{k\alpha}(0) < k(m_2 - m_1)$$

Now, we consider the rotation number

$$\rho = \lim_{k\to\infty} \frac{F^{k\alpha}(0) - 0}{k\alpha} \leqslant \lim_{k\to\infty} \frac{k(m_2 - m_1)}{k(n_1 - n_2)} = \frac{m_2 - m_1}{n_1 - n_2}$$

Rearranging, we reach the desired conclusion

$$n_1\rho + m_1 < n_2\rho + m_2$$

where we have omitted the case of equality because it leads to a contradiction. Notice that the other direction is already proven because it involves the case where the inequality is reversed. We have

$$F^{n_1}(x) + m_1 > F^{n_2}(x) + m_2 \implies n_1\rho + m_1 > n_2\rho + m_2$$

by an analogous argument. ■

Lemma about ω-limit sets. Take the same setting as the previous lemma. The ω-limit set of any point x does not depend on x and only on the map f. Under these conditions, we must have $\omega(x) = S^1$ or $\omega(x)$ as a Cantor set (i.e. compact set with empty interior and no isolated points)

Proof. First, we explain why there is no dependence on x. Consider a point x and its orbit. Pick some distinct integers n, m and fix an interval I between the points $f^n(x), f^m(x)$; we fix an interval because on a circle there is a choice between two intervals. Let y be another point. We claim that there exists some integer k such that $f^k(y) \in I$. Define an interval

$$I_k = f^{-k(n-m)}(I)$$

Consider the endpoints of I, which are $f^n(x)$ and $f^m(x)$. When we apply the map $f^{-(n-m)}$ to both of these points, we have $f^n(x) \mapsto f^m(x)$ and $f^m(x) \mapsto f^{2m-n}(x)$. So, I_1 will be between these two image points. Subsequent applications of this map $f^{-(n-m)}$ have similar behavior. Hence, all of the intervals $I, I_1, I_2, \ldots$ share a boundary point. We claim that

$$S^1 = \bigcup_{k\in\mathbb{N}} I_k$$

Suppose, for contradiction, that these intervals I_k does not cover the circle. Then, there is some interval (Why?) that is not covered by the union $\bigcup_k I_k$. Also, denote the left and right endpoints of I_k as

$$l_k := f^{-k(n-m)}(f^n(x)), \quad r_k = f^{-k(n-m)}(f^m(x))$$

The sequence $\{r_k\}$ is a subset of the circle, which is compact, so there exists a subsequence $r_{k_j} \xrightarrow[j\to\infty]{} z$. Observe that $f^{n-m}(r_{k_j+1}) = r_{k_j}$. By continuity, we have

$$z = \lim_{j\to\infty} r_{k_j} = \lim_{j\to\infty} f^{n-m}(r_{k_j+1}) = f^{n-m}\left(\lim_{j\to\infty} r_{k_j+1}\right) = f^{n-m}(z)$$

This shows that z is a periodic point, a contradiction. Therefore, the unions of the intervals I_k cover S^1. From this, we see that there exists some $k \in \mathbb{N}$ such that $y \in I_k$. But then taking the inverse function, we have $f^{k(n-m)}(y) \in I = [f^n(x), f^m(x)]$, our original claim. Now, what does this claim say about $\omega(x)$? Take $a \in \omega(x)$. By the definition of ω-limit set, there exists a sequence $f^{n_k}(x) \to a$. Consider the two points $f^{n_k}(x), f^{n_k+1}(x)$. By our previous statement, there exists some $f^{p_k}(y)$ which is contained in the interval between these two points. But then $f^{p_k}(y) \xrightarrow[k\to\infty]{} a$. Thus, $a \in \omega(y)$. Since x and y were arbitrary, this shows that $\omega(x) = \omega(y)$ and so there is no dependence of the ω-limit set on any point.

We now move to proving the assertions about what $\omega(x)$ is. Call $\omega(x) = E$. Since it is an ω-limit set, we know that E is compact and invariant. Assume that we may pick a point $b \in \partial E$, the boundary of E. Since f is a homeomorphism, ∂E is also invariant. It follows that $f^n(b) \in \partial E$ and this implies that $\omega(b) \subseteq \partial E$. From our above discussion, we know that $\omega(b) = E$ and we have shown that $E \subseteq \partial E$ with the preceding statement. Hence $E = \partial E$. In this case, we see that E is a Cantor set with a finite number of isolated points. Suppose that x is a point of E. This means that x is an element of $\omega(x)$. So, there must be a sequence $f^{n_k}(x)$ which tends to x, and none of these points can coincide (otherwise there is a periodic orbit). But then x is not isolated. So there are no isolated points in the ω-limit set.

Assume it is not possible to pick a point from ∂E, that is $\partial E = \varnothing$. In this case, $E = S^1$. ■

Theorem. *Poincaré Classification Theorem*

Let f be an orientation-preserving circle diffeomorphism. Let the rotation number $\rho(f)$ be irrational. Exactly one of the following occurs:

1. f has a dense orbit if and only if $f \sim R_\rho$ (topologically conjugated to a rotation).

2. There exists h such that f is semiconjugate to a rotation, that is

$$h \circ f = R \circ h$$

where h is continuous and monotonic.

Remark. Thinking of the lemma on ω-limit sets, we might see that the first case corresponds to when the ω-limit set is the entire circle and the second case corresponds to when it is a Cantor set.

Proof. Let F be a lifting of f. We consider the full preimage of the orbit

$$B = \{F^n(x) + m\}_{n,m\in\mathbb{Z}}$$

Define a map $H : B \to \mathbb{R}$ by

$$F^n(x) + m \mapsto n\rho + m$$

Notice that H is monotonic. Is H well-defined? From our first lemma, we know that

$$F^{n_1}(x) + m_1 < F^{n_2}(x) + m_2 \implies n_1\rho + m_1 < n_2\rho + m_2$$

So strict monotonicity holds. The point of taking H is because it is a conjugacy. Observe that

$$H \circ F = R_\rho \circ H$$

on B. Assume that F is the canonical lifting which maps the unit interval to the unit interval. We may directly compute that

$$(H \circ F)(F^n(x) + m) = H(F^{n+1}(x) + m) = (n+1)\rho + m$$
$$(R_\rho \circ H)(F^n(x) + m) = \rho + n\rho + m$$

This shows the expressions are equal. Our goal is to show that this equality holds not only on B but also on $\mathbb{R}$. Let us consider continuing the map to the closure $\overline{B}$. If $y = \lim_{n\to\infty} x_n$ where $x_n \in B$, we define

$$H(y) = \lim_{n\to\infty} H(x_n)$$

How do we know that this limit exists? We can consider splitting up the sequence terms into those which are above y and those below y, if we can prove that these terms converge to the same limit on both sides, we can obtain the correct continuation on the closure of B. Now, there is still a problem if $I \subseteq \mathbb{R}\backslash H(B)$. In this case, we will have a discontinuity. Suppose, for contradiction, that such an I exists. Recall that we map into the set $\{n\rho + m\}_{n,m\in\mathbb{Z}}$. Recall that for irrational rotation on a circle, all orbits are dense. Consider the projection into the circle $\pi(I) \subseteq S^1$. There is some point $n\rho \in \pi(I)$ because of its density. As a result, there exists a corresponding point $n\rho + m \in I$. This contradicts the construction of I as a set in $\mathbb{R}\backslash H(B)$. Thus, this continuation is possible on $\overline{B}$.
We have the map H on $\overline{B}$.

1. Suppose that a dense orbit exists. We may pick a point x that has a dense orbit. Then, our H will be strictly monotone and extend to the domain $\mathbb{R} = \overline{B}$. It turns out that H is a homeomorphism.

2. Suppose that there is no dense orbit. We will have a function on $\overline{B}$ with horizontal gaps, where the function will have the same value on either side of the gap. Recall that vertical gaps are impossible due to the argument we gave above. It is straightforward to extend the function continuously on these gaps by drawing a horizontal line (i.e. set all values of the function between the gaps equal to the values on the sides of the gaps). We will have a monotone function that is not strictly monotone.

Further

$$H \circ F = R_\rho \circ H$$

on $\mathbb{R}$. Observe that

$$H(Z+1) = H(F^n(x)+m+1) = n\rho + m + 1 = H(Z)+1$$

This holds on $\mathbb{R}$ by the continuous extensions we have used (applies for either case). Hence, we may project this function onto the circle and obtain

$$h \circ f = R_\rho \circ h$$

This is the conclusion we desired. ■

Remark. We are interested in when these cases are possible. For $\mathscr{C}^1$, there is a Denjoy counterexample where the second case is realized (i.e. this shows why smoothness is necessary). In $\mathscr{C}^2$, we have the conjugacy mentioned in the first case (which actually follows from distortion control and not rotation number).

18.2 Structural Stability and Connected Topics

Proposition. The rotation number $\rho(\cdot)$ is continuous on $\text{Diff}_+(S^1)$ (the orientation-preserving diffeomorphisms of the circle).

Proof. Assume that we have a map f with rotation number $\rho(f)$ and we find two rational numbers so that

$$\frac{p'}{q'} < \rho(f) < \frac{p}{q}$$

The intuitive idea is that we want to shrink this interval. For a lift F, we claim that $F^q(x) - x < p$. We may write

$$\frac{F^n(x)-x}{n} = \sum_{i=1}^{n} \frac{F^i(x) - F^{i-1}(x)}{n} \geqslant \min F(y) - y$$

If we take $n = qk$, then

$$\frac{F^n(x)-x}{n} = \sum_{i=1}^{k} \frac{F^{qi}(x) - F^{q(i-1)}(x)}{qk} \geqslant \min_y \frac{F^q(y)-y}{q}$$

Assume, for contradiction, that $F^q(x) - x > p$. Since the rotation number does not depend on a point, we may choose y instead of x. Dividing through by q, we have $\frac{F^q(y)-y}{q} > \frac{p}{q}$. Hence

$$\rho(f) = \lim_{n\to\infty} \frac{F^n(y)-y}{n} \geqslant \min_y \frac{F^q(y)-y}{q} > \frac{p}{q}$$

This contradicts our initial assumptions. A similar argument shows that we cannot have $F^q(x) - x = p$. For some $\delta > 0$, we may write

$$F^q(x) - x < p - \delta$$

Suppose that we choose $\text{dist}(f,g) < \varepsilon$ for sufficiently small $\varepsilon > 0$. As a result, we can make $\text{dist}(F^q, G^q) < \frac{\delta}{2}$. We may obtain the same inequality for G now, that is $G^q(x) - x < p$. Hence $\rho(g) < \frac{p}{q}$ using a similar argument involving the average estimate and where we bound $\rho(g)$ above by a maximum.

$$\rho(g) = \lim_{n\to\infty} \frac{G^n(y) - y}{n} \leqslant \max_y \frac{G^q(y) - y}{q} < \frac{p}{q}$$

Thus, we see that $\rho(f), \rho(g) \in \left[\frac{p'}{q'}, \frac{p}{q}\right]$. To conclude continuity, let a be the distance between the endpoints of this interval and let $b = \varepsilon$. It follows that for every a, there exists a b such that $\text{dist}(f,g) < b$ implies that $|\rho(f) - \rho(g)| < a$. ■

Remark. This continuity appears different when we take rational rotation numbers and when we take irrational rotation numbers. Roughly speaking, suppose we have a map f with an irrational rotation number. We can find another map $\tilde{f}$ that is in some neighborhood U of f such that $\rho(f) \neq \rho(\tilde{f})$. But since rotation number is a topological invariant, it follows that f and $\tilde{f}$ cannot be topologically conjugated. Therefore, f is not structurally stable.

Now, we discuss the criteria of structural stability for a circle diffeomorphism.

Theorem. *Structural Stability for a Circle Diffeomorphism*
Let f be an orientation-preserving circle diffeomorphism. Suppose that the following conditions hold.

1. All periodic orbits are hyperbolic, that is $(f^q)' \neq 1$.
2. There exists at least one periodic orbit.

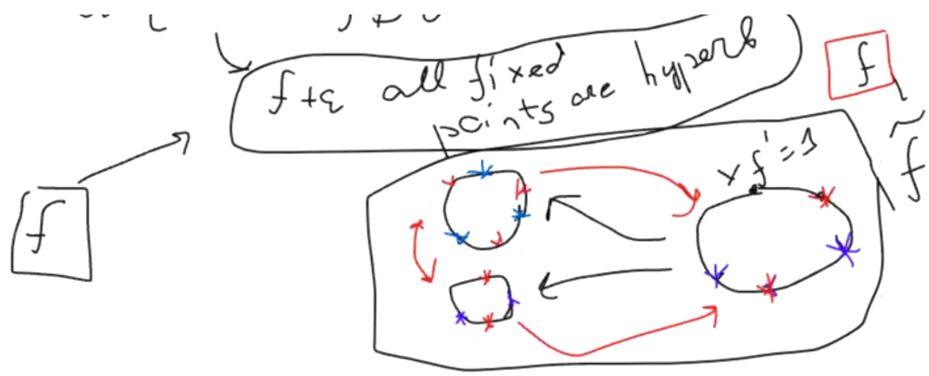
These conditions are equivalent to structural stability.

Recall that there is a global condition for structural stability which involves the two conditions: Axiom A and strong transversality. Let us consider how the conditions in our criteria above might correspond. For Axiom A, we requires a non-wandering set which is hyperbolic for which periodic points form a dense set. This condition matches the two given in the above theorem already. Strong transversality means that stable and unstable manifolds intersect transversely. However, on 1-dimensional manifolds, all intersections are transversal so that this condition is trivially satisfied.

Proof Sketch. To see sufficiency, note that it can be shown that there are a finite number of fixed points and that attracting and repelling points must alternate. Now, the dynamics on each arc of the circle are identical to those of the North-South map (recall that the NS map is structurally stable). It is possible to glue these local conjugacies to get a global homeomorphism.

For necessity, suppose that f is structurally stable. Then, the rotation number $\rho(f)$ must be rational or irrational. If $\rho(f)$ is irrational, then it is not structurally stable, so this case is not possible. Hence $\rho(f)$ is rational. Previously, we have shown that $\rho(f) \in \mathbb{Q}$ means that a periodic orbit exists. Now, consider the function $F - \mathrm{Id}$, where F is a lifting of f. Recalling Sard's lemma, we know that the set of critical values of $F - \mathrm{Id}$ has zero measure. A critical value of $F - \mathrm{Id}$ occurs when $(F - \mathrm{Id})' = 0$, or $F' = 1$ (which is equivalent to $f' = 1$). From Sard's lemma, there exists an $\varepsilon > 0$ such that 0 does not belong to the set of critical values of $F + \varepsilon - \mathrm{Id}$. Thus, there exists an $f + \varepsilon$ such that all fixed points are hyperbolic.

We note that it is possible to perturb our map f to obtain $\tilde{f}$ so that we have all hyperbolic points and one parabolic point (satisfies $f' = 1$). In this case, we may split the parabolic point into two hyperbolic points by a perturbation or remove it altogether. The end result is a contradiction if we require that these maps are conjugated to each other, as they will have a different number of fixed points. This idea is shown in the image below.



To summarize the approach:

1. We begin with a map f which has periodic orbits, of which some points in the orbit may be parabolic.

2. If the number of orbits is infinite, we may perturb the map slightly and obtain a map with all isolated hyperbolic points and finite number of periodic orbits. There cannot be a conjugacy in this case.

3. If there are parabolic points but a finite number of orbits, we can apply the argument we used in the picture.

Ask for this argument to be redone. It is a bit unclear in its presentation.

19 April 30, 2025

19.1 Structural Stability and Circle Diffeomorphisms

Some of the material in the previous lectures may be cross-referenced in Katok and Hasselblatt, on P. 387-410. We return to our discussion of maps on a circle. Let f be an orientation-preserving diffeomorphism of the circle. Recall that we need two conditions for structural stability:

1. There exists at least one periodic orbit.
2. Every periodic point is a hyperbolic point. In this simple context, this condition means
$$(f^q)'(p) \neq 1$$
for some minimal period q.

These two conditions are equivalent to structural stability. We have already discussed the reasons for why these conditions are sufficient for structural stability. The intuitive reason is that if these conditions are satisfied, then we have a map that is similar to the North-South map, which we know is structurally stable. We may essentially repeat the argument, but for more points, noting that there must be a *finite number* of periodic points if all periodic points are hyperbolic.

Now, why are these conditions also necessary for structural stability? This is similar to why we have all hyperbolic points in Andronov-Pontryagin theorem (i.e. use Sard's lemma and proceed in the same way as AP). This is why the second condition is necessary.

On a naïve level, if we have a parabolic point, we can destroy it and obtain two points, resulting in two maps that are not conjugated. This shows again why the condition of hyperbolicity is important. To be more explicit:

- Destroy means to perform a small $\mathscr{C}^1$ perturbation that removes the parabolic point.
- Obtaining two points refers to the phenomenon that generic perturbations create two hyperbolic points: a source and a sink.
- The number and type of periodic points are invariant under conjugacy.

Returning to the conditions, why is the first condition necessary? We claim that diffeomorphisms with no periodic points are not structurally stable.

Proposition. Diffeomorphisms with no periodic points are not structurally stable.

Proof. Recall the definition of rotation number. Let $\rho(F)$ denote the rotation number of F. We first prove a lemma. If $\rho(F) \in \mathbb{R}\backslash\mathbb{Q}$, then f is not structurally stable. We begin with some map f with irrational rotation number and we consider the standard lifting, F. We adopt the following definition for order, we say that $f_1 < f_2$ if $F_1 < F_2$ in some neighborhood in the space of diffeomorphisms. Note that we want to avoid $\hat{F}_1 = F_1 + 1$, because then we may run into the contradiction $f_1 < f_2 < f_1$ (because $\hat{F}_1$ and F_1 both correspond to f_1). Thus, we should only add sufficiently small ε.

Let f_1 and f_2 be two diffeomorphisms on the circle satisfying $f_1 < f_2$, as above. Now, we would like to show that for sufficiently small κ, that if we have f_1 and $f_1 + \kappa$ (where $f_1 + \kappa$ exists within a small neighborhood of f_1), then $F_1 < F_1 + \kappa \implies \rho(f_1) < \rho(f_1 + \kappa)$. In order to prove this, we need to first show another statement: If $\rho(f_1) \in \mathbb{R}\backslash\mathbb{Q}$, then $\rho(f_1) < \rho(f_2)$. However, for now, we will accept that $F_1 < F_1 + \kappa \implies \rho(f_1) < \rho(f_1 + \kappa)$ (<u>Justify why the proposition yields this</u>). Recall that rotation numbers are topologically invariant (i.e. $\rho(hfh^{-1}) = \rho(f)$). Hence, if f_1 and $f_1 + \kappa$ are topologically conjugated, then they must have the same rotation number. As a result, we conclude that f_1 and $f_1 + \kappa$ are *not* topologically conjugated. Hence, this shows that in any neighborhood of f_1, there exists a map that is not topologically conjugated to it. Therefore, f_1 is not structurally stable.

Remark. Structural stability depends on the space in which we work. It may be possible that our map is not structurally stable in the space of 1 times smooth diffeomorphisms, but is structurally stable for 10 times smooth diffeomorphisms. But, adding κ means that our new map will be in the same class as our original map, so this argument holds regardless of the space.

At this point, we return to proving our earlier statement.

Proposition. If $f_1 < f_2$ and $\rho(f_1) \in \mathbb{R}\backslash\mathbb{Q}$, then $\rho(f_1) < \rho(f_2)$.

Proof. Since $f_1 < f_2$, we have $F_2 - F_1 > 0$. Observe that $F_2 - F_1$ must be $\mathbb{Z}$-periodic and continuous. So, there exists a minimum $F_2 - F_1 > \delta > 0$. Here, we take $\delta = \min_{x \in \mathbb{R}}(F_2 - F_1)$, which exists because $F_2 - F_1$ attains a minimum and maximum on the compact interval $[0, 1]$. We would like to estimate our rotation number between two values:

$$\frac{p - \delta}{q} < \rho(F_1) < \frac{p}{q}$$

We consider and eliminate possible cases.

1. Suppose $F_1^q(y) - y < p - \delta$ for all y. Then we have

$$\lim_{n \to \infty} \frac{F_1^{qn}(x) - x}{qn} = \lim_{n \to \infty} \sum_{i=1}^{k} \frac{F_1^q(F_1^{q(i-1)}(x)) - F_1^{q(i-1)}(x)}{qn}$$

   Now, if we take $y = F_1^{q(i-1)}(x)$, we have

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{F_1^q(y) - y}{qn} \leqslant \lim_{n \to \infty} \sum_{i=1}^{n} \frac{p - \delta}{qn} = \frac{p - \delta}{q}$$

Therefore, $\rho(f_1) \leqslant \frac{p-\delta}{q}$. This is a contradiction, so this situation cannot happen.

2. There exists an x_0 such that $F_1^q(x_0) - x_0 > p - \delta$. Consider that

$$\begin{aligned} F_2^q(x_0) &= F_2(F_2^{q-1}(x_0)) > F_1(F_2^{q-1}(x_0)) + \delta \\ &> F_1^q(x_0) + \delta > x_0 + p \end{aligned}$$

where we have used $F_2^{q-1}(x) > F_1^{q-1}(x)$ and monotonicity. Now, there are two more subcases.

(a) We have $F_2^q(y) > x_0 + p$ for all y. If we proceed as we have in (1), then we will see that $\rho(f_2) \geqslant \frac{np}{nq} = \frac{p}{q}$.

(b) There exists some x_1 such that $F_2^q(x_1) \leqslant x_1 + p$. By continuity and monotonicity, there is some x_e such that $F_2^q(x_e) = x_e + p$. To give more detail, we have a point x_0 such that $F_2^q(x_0) > x_0 + p$ and a point x_1 such that $F_2^q(x_1) \leqslant x_1 + p$, and the intermediate value theorem shows that x_e such that $F_2^q(x_e) = x_e + p$ exists.
Thus, when we project this function onto the circle, we have $f_2^q(x_e) = x_e$. This shows that $\rho(f_2) = \frac{p}{q}$. To be clear, made this conclusion from

$$\lim_{n\to\infty} \frac{F_2^{qn}(x_e) - x_e}{qn} = \lim_{n\to\infty} \frac{np}{nq} = \frac{p}{q}$$

Summarizing the cases (a) and (b), we see that $\rho(f_2) \geqslant \frac{p}{q} > \rho(f_1)$. This proves our proposition.

□

Now, because this proposition holds, our outer proposition about structural stability holds as well. ■

19.2 Denjoy's Theorem and Denjoy's Counterexample

Despite the fact that maps with irrational rotation numbers are not structural stable, they have more attractive dynamical properties.

Theorem. *Denjoy's Theorem*
Let $f : S^1 \to S^1$ be an orientation-preserving diffeomorphism of the circle. If $f \in \mathscr{C}^2$ and $\rho(f) \notin \mathbb{Q}$, then f is topologically conjugated to a rotation $R_{\rho(f)}$.

Denjoy's Counterexample. This counterexample is related to the theorem above. It turns out that if $f \in \mathscr{C}^1 \backslash \mathscr{C}^2$, then there can be no conjugacy.

Remark. This result is applicable in mechanics and complex dynamics.

Remark. It is useful to keep in mind the pattern suggested by Denjoy's theorem. That is given a map which satisfies certain properties, we are able to conclude that this map is topologically conjugated to a standard map. We have seen this pattern before when we discussed circle doubling. We saw that any map with degree 2 that is close enough to circle doubling is conjugated to circle doubling. This approach is known as finding **normal forms** of maps or flows. This is an important approach because instead of working with maps and their properties, we are able to equivalently work with a standard map, of which we have a better understanding.

Proof of Denjoy's Theorem. Recall that we proved the Poincaré classification theorem. If a map f (which only has to be a homeomorphism) has a rotation number which is irrational, $\rho(f) \notin \mathbb{Q}$, then there are two possibilities. First, it is conjugated to a rotation, $f \sim R_\rho$. Second, it is a semi-conjugacy $f \circ h = h \circ R_\rho$.

Aside. Recall that the meaning of semiconjugacy. Let $f : X \to X$ and $g : Y \to Y$ be two dynamical systems. g is semiconjugate to f if there exists a continuous and surjective map $h : Y \to X$ such that

$$h \circ g = f \circ h$$

Another way to understand semiconjugacy is to say that the diagram below commutes

$$\begin{array}{ccc} Y & \xrightarrow{\quad g \quad} & Y \\ \big\downarrow h & & \big\downarrow h \\ X & \xrightarrow{\quad f \quad} & X \end{array}$$

We want to exclude the case of semi-conjugacy. To do this, we rely on a technique called distortion control. We need $f \in \mathscr{C}^{1+\mathrm{Var}}$. This notation means f' has a bounded variation.

Definition. *Variation*

The variation of any function g, may be given as

$$\mathrm{Var}(g) = \sup_{n, I_k} \left\{ \sum_{k=1}^{n} |g(x_k) - g(x_k')| \right\}$$

where we require x_k, x_k' to be the endpoints of some interval I_k and two intervals $I_k \cap I_j = \varnothing$ (disjointness). So, we split our interval into finitely many segments and calculate this sum. The variation may be bounded or unbounded.

It is possible for continuous maps to have unbounded variation. However, it may be shown that any $\mathscr{C}^2$ map (on a compact set) has bounded variation.

Returning to our argument, from the Poincaré classification theorem, there are two possibilities for our map. If our map is conjugated to a rotation we are done. So suppose that it is a semi-conjugacy. Previously, we have shown that when f is semi-conjugated to a rotation, then the ω-limit sets of any point, say x, belong to the same Cantor set E. Suppose that $I \subseteq S^1 \backslash E$ is an interval. The images and preimages of I do not intersect - it never returns to itself. Now, we compute

$$\begin{aligned}\ell[f^n(I)] + \ell[f^{-n}(I)] &= \int_I (f^n)'(x) + (f^{-n})'(x)\, dx \\ &\geqslant \int_I \sqrt{(f^n)'(x) \cdot (f^{-n})'(x)}\, dx\end{aligned}$$

where the inequality follows from the AM-GM inequality. At this point, we use distortion control. We have $\mathrm{Var}(f') \sim V$. We continue

$$\begin{aligned}\int_I \sqrt{(f^n)'(x) \cdot (f^{-n})'(x)}\, dx &\geqslant \int_I \sqrt{e^{-V}}\, dx \\ &= \ell(I) \cdot e^{-\frac{V}{2}}\end{aligned}$$

Note that this inequality from distortion control holds for infinitely many n, but not all n. So, we have

$$\sum_{i=-\infty}^{\infty} \ell(f^i(I)) = \infty > 1 = \ell(S^1)$$

But this gives us $f^n(I) \cap f^n(I) \neq \varnothing$, which contradicts the Poincaré classification theorem.

■

19.3 Distortion Control

We elaborate on the technique of distortion control, which was given without justification in the previous proof. Use the same notation and setting in the proof and let $J = [x, y]$ be an interval. Consider the interval and its images

$$J, f(J), \ldots f^n(J), \ldots$$

which all have disjoint intersection. We will also let $J^k = [f^k(x), f^k(y)]$. Consider the function $\varphi(x) = \ln|f'(x)|$. The composition of two functions of bounded variation is again a function of bounded variation. Hence $\text{Var}(\varphi) = V < \infty$. Now, we estimate this variation

$$\begin{aligned} V &\geqslant \sum_{k=0}^{m-1} |\varphi(f^k(x)) - \varphi(f^k(y))| \geqslant \left| \sum_{k=0}^{m-1} \varphi(f^k(x)) - \varphi(f^k(y)) \right| \\ &= \left| \ln \left(\prod_{k=0}^{m-1} f'(f^k(x)) \prod_{k=0}^{m-1} (f')^{-1}(f^k(y)) \right) \right| \\ &= \ln \left| \frac{(f^m)'(x)}{(f^m)'(y)} \right| \end{aligned}$$

where the last equality comes from the chain rule. Exponentiating the first and last expression in the chain, we have

$$\left| \frac{(f^n)'(x)}{(f^n)'(y)} \right| < e^V$$

The same calculation also gives

$$e^{-V} < \left| \frac{(f^n)'(x)}{(f^n)'(y)} \right| = \left| (f^n)'(x)(f^{-n})'(x) \right|$$

For the last equality, recall from the inverse function theorem (or chain rule) that

$$(f^{-n})'(x) = \frac{1}{(f^n)'(y)}$$

where $y = f^{-n}(x)$. Now, recalling that $J = (x, y)$ and we claimed that $f^m(J) \cap f^t(J) = \varnothing$. For our intervals I, we know that this is the case, but we have not shown this for intervals of the form $(x, f^{-n}(x))$, where $0 \leqslant m, t < n$. Resume at 1:12:00, Lecture 19

20 April 30, 2025

20.1 Complex Dynamics

Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. While we do not assume this is a diffeomorphism, it is still possible to look at positive orbits $\{z, f(z), \ldots, f^n(z), \ldots\}$. There are similarities of complex dynamics to both 1-dimensional and 2-dimensional dynamics because of the structure of the complex plane. We are usually interested in the case when our function is a polynomial rational function, or an entire function (i.e. $\sin, \cos, \exp$).

Today, we focus on the case of the dynamics of rational functions. For instance, we can consider $f(z) = z^2$ on the Riemann sphere $S^2 = \hat{\mathbb{C}}$. We note that there are two fixed points, 0 and ∞. For all numbers where $|z| < 1$, we see that trajectories are going to 0, for all numbers where $|z| > 1$, trajectories are going to ∞. For $|z| = 1$, we have the map $(1, \varphi) \mapsto (1, 2\varphi)$.
In summary, this map has two attracting points and a circle where on the circle there is circle doubling.

20.2 Lyapunov and Asymptotic Stability

We discuss a new concept that is important for analysis of complex dynamics: pointwise stability. This is a different concept from structural stability. This type of stability applies to sets and points.
Let f be continuous. Let X be a compact, invariant set under f with a metric. We define the neighborhood of a set X as

$$U_\varepsilon(X) := \{y \in M : d(y, X) < \varepsilon\} \quad \text{where } d(y, X) = \inf_{x \in X} d(y, x)$$

Definition. *Stability of a Space*
We say that X is **Lyapunov stable** if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $n \in \mathbb{N}$, we have $f^n(U_\delta(X)) \subseteq U_\varepsilon(X)$.
We say X is **asymptotically stable** if it is Lyapunov stable and there is some $\delta' > 0$ so that every a with $d(a, X) < \delta'$ actually converges to X :

$$\lim_{n \to \infty} d\left(f^n(a), X\right) = 0$$

Equivalently,

$$\exists \delta' > 0 \quad \text{such that} \quad a \in U_{\delta'}(X) \Longrightarrow \omega(a) \subseteq X,$$

Definition. *Stability of a Point*
A point x in X is said to be **Lyapunov stable**, if,

$$(\forall \epsilon > 0)\,(\exists \delta > 0)\,(\forall y \in X)\,\left[d(x,y) < \delta \Rightarrow (\forall n \in \mathbb{N})\, d\left(f^n(x), f^n(y)\right) < \epsilon\right].$$

We say that x is **asymptotically stable** if it belongs to the interior of its stable set, i.e. if,

$$\exists \delta > 0 \left[d(x,y) < \delta \Rightarrow \lim_{n\to\infty} d\left(f^n(x), f^n(y)\right) = 0\right]$$

Example. Using the function $f(z) = z^2$ on the Riemann sphere and the spherical metric, we see that all trajectories on the inside or outside of the unit circle are asymptotically stable (they approach 0 or ∞). Under these dynamics, the Riemann sphere splits up into the Fatou set which consists of points outside the unit circle and the Julia set which consists of the unit circle.

Definition. *Fatou set, Julia set*
The Fatou set $F(f)$ is the union of all points with Lyapunov stable orbits under f. The Julia set is simply the complement of the Fatou set $J(f) = \hat{\mathbb{C}} \backslash F(f)$, equivalently it is the set of all unstable orbits.

Remark. It is easy to see that the unit circle contains the points with unstable trajectories (i.e. the Julia set) under $f(z) = z^2$ because circle doubling is mixing.
Here are some properties of the Fatou and Julia sets.

1. The Fatou set is open while the Julia set is closed.
2. The Julia set is always nonempty. The Fatou set is nonempty for polynomials. To see this, we note that all polynomials look like their highest order powers near infinity. So, disks 'near' infinity are sent to smaller disks.
3. In fact, it may be shown for any rational function that the Fatou set is nonempty.
4. However, it is possible for the Fatou set to be empty. For instance, for the exponential function the Julia set is the entire complex plane; $J(e^z) = \hat{\mathbb{C}}$

Remark. The family of functions $f(z) = z^2 + c$, where c is some parameter, is the subject of many famous visualizations (see Julia sets of these). For instance, the Douady rabbit, Mandelbrot set, and so forth. These often involve connected Julia sets.

Theorem. *About Julia Set of Polynomials*
For a polynomial, the Julia set is connected if and only if the orbit of all critical points

do not tend to infinity. If the Julia set is not connected, then there are uncountably many components.

Remark. We spoke a bit about the intuition behind the fractal structure of the Mandelbrot set here (at 23:00).

20.3 Normal Forms

With the same setting for f, suppose we have a periodic point $f^p(z) = z$. For simplicity, we begin with a function with a fixed point at 0, that is $f(0) = 0$.

Definition. *Types of Fixed Points*
We classify fixed points.

1. If $0 < |f'| < 1$, we call the point an **attracting point**. If $f' = 0$, we call the point a **superattracting point**. These points and their neighborhoods belong to the Fatou set.
2. If $|f'| > 1$, we call the point a **repelling point**. These points belong to the Julia set.
3. If a point is either a repelling point or an attracting point, it belongs to the **hyperbolic case**.
4. If we have $|f'| = 1$, this is called the **neutral case.** In our standard real dynamics, this is analogous to the parabolic case. In this case, the point may belong to either the Julia or Fatou sets.

We move to a description of normal forms.
Notation. We are using expressions like $f(z) = \lambda z + \dots$ to abbreviate higher-order terms of a Taylor series expansion of f around the fixed point. So

$$f(x) = \sum_{n=1}^{\infty} a_n z^n = a_k z^k + a_{k+1} z^{k+1} + \dots$$

where k is the index of the first nonzero coefficient.

I. Suppose we have a map f given by $z \mapsto \lambda z + \dots$ which corresponds to the **hyperbolic case** (i.e. $0 < |\lambda| < 1$, or $|\lambda| > 1$). This map is conjugated analytically to the map $z \mapsto \lambda z$. This is called the **Poincaré case**. It turns out the homeomorphism in this case is

$$h = \lim_{n\to\infty} \lambda^{-n} f^n$$

Observe that

$$h \circ f = \lim_{n\to\infty} \lambda^{-n} f^{n+1} = \lim_{n\to\infty} \lambda \cdot (\lambda^{-(n+1)}) f^{n+1} = \lambda h$$

showing conjugacy. It may also be shown that h is a local diffeomorphism and holomorphic as well.

II. The **supercritical or superattracting case**. It is associated with Böttcher. Suppose we have a map $f(z) = z^k + \ldots$ where $k > 1$. We omit the coefficient because by an affine transformation we eliminate any coefficient (conjugate f by a scaling map).

> *Aside.* A linear change of coordinate (an affine map with the fixed point already at 0) is simply a scaling
>
> $$\Phi_c(w) = cw, \quad c \in \hat{\mathbb{C}}$$
>
> Conjugating f by Φ_c gives
>
> $$\begin{aligned}\tilde{f} &:= \Phi_c^{-1} \circ f \circ \Phi_c(w) = c^{-1} f(cw) \\ &= c^{-1}\left[a_k (cw)^k + a_{k+1}(cw)^{k+1} + \cdots\right] = a_k c^{k-1} w^k + a_{k+1} c^k w^{k+1} + \cdots\end{aligned}$$

This map is conjugated to $z \mapsto z^k$. The conjugate map is

$$h = \lim_{n\to\infty} \sqrt[k^n]{f^{\circ n}}$$

Why is this map holomorphic?

III. **Diophantine numbers.** The map $f(z) = e^{2\pi i\alpha} z + \ldots$ is conjugated to the map Λ given by $z \mapsto e^{2\pi i\alpha} z$ when α is Diophantine. This third normal form is also known as the **Siegel (or small–divisor) linearization theorem**.

> **Definition.** *Diophantine Number*
> We say that a real number α is (c, d)-Diophantine if
>
> $$\forall p \in \mathbb{Z}, q \in \mathbb{Z}_+ : |q\alpha - p| > cq^{-d}$$
>
> We call a number Diophantine when there exists some c, d for which it is (c, d)-Diophantine. There is another common form of the definition when we divide the inequality through by q.

Remark. The point of saying a number is Diophantine is to get a rough measure of its irrationality. A number which is Diophantine may be thought of as "very irrational".

Definition. *Liouville Number*
A real number α is a Liouville number if for every integer $n \geqslant 1$ there are infinitely many rationals p/q with

$$0 < \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^n}$$

Equivalently, α can be approximated by rationals with error smaller than any power of the denominator.

Remark. Rational and Liouville numbers are not Diophantine. Liouville numbers are a counterpart to Diophantine number in that they are "too-well" approximable. Liouville numbers form a dense G_δ set of Lebesgue measure zero, and in particular every Liouville number is transcendental.

20.4 Kolmogorov-Arnold-Moser Theory

We abbreviate this as KAM theory. There is a connection between almost all algebraic structures unified after KAM theory. This theory has interesting applications, such as explaning why the solar system is stable under small perturbations. For instance, one asteroid could not destroy the stability of the solar system (many invariant curves are persistent).

To begin our discussion, we give an analogy with Newton's method. The goal of Newton's method usually is as a root-finding algorithm. Usually, Newton's method involves iterating the following:

1. Pick a point near the root and find the linear approximation (tangent line) at that point.
2. Consider the intersection of this tangent line with the x-axis and use this as the new point of a subsequent approximation.

Consider a modification to Newton's method where at any point, we use the tangent line to zero rather than the tangent line to that point. Will we have faster convergence. Our main goal is superexponential convergence.

Now, we begin introducing a new method. Recall what we have discussed already. We had a method involving the inverse function theorem when we proved the persistence of periodic hyperbolic points. Hyperbolic fixed points are related to conjugacy through the Grobman-Hartman theorem. We had a method of contractions when we proved structural stability of circle doubling. This is a third method to obtain conjugacy. We start with the following function of two arguments

$$\mathcal{F}(f, h) = h^{-1} \circ f \circ h$$

Our goal is to solve the equation

$$\mathcal{F}(f, h) = g$$

that is, we want to show that f and g are conjugated. Observe that if we have $f = g$, there is the trivial solution $F(g, \mathrm{Id}) = g$. Our idea is this: considering (g, Id) as a point in the two argument space, we want to make an approximation of $\mathcal{F}$ near this point. We have an Taylor approximation

$$\mathcal{F}(f, h) = \mathcal{F}(g, \mathrm{Id}) + D_1\mathcal{F}(g, \mathrm{Id})(f - g) + D_2\mathcal{F}(g, \mathrm{Id})(h - \mathrm{Id}) + R((g - f), (h - \mathrm{Id}))$$

where R indicates a remainder in higher order terms, and where D_1, D_2 indicate the derivative in the first and second arguments, respectively. Ignoring the remainder (it seems like we change h to h_1 to account for this) and considering the initial equation we would like to solve, we have the equation

$$\mathcal{F}(g, \mathrm{Id}) + D_1\mathcal{F}(g, \mathrm{Id})(f - g) + D_2\mathcal{F}(g, \mathrm{Id})(h_1 - \mathrm{Id}) = g$$

Solving this yields

$$f_1 = h_1^{-1} \circ f \circ h_1$$

We hope that since we deleted only a term of second order (the remainder R), the result of solving this equation will be correct up to the second order. Putting this as a condition, we expect

$$|f_1 - g| = O((f - g)^2)$$

By continuing this procedure, we obtain our final map

$$h = \lim_{n \to \infty} h_1 \circ h_2 \circ \cdots \circ h_n$$

With this plan in mind, we begin to solve the equation involving the approximation.

$$\begin{aligned} \mathcal{F}(g, \mathrm{Id}) + D_1\mathcal{F}(g, \mathrm{Id})(f - g) + D_2\mathcal{F}(g, \mathrm{Id})(h_1 - \mathrm{Id}) &= g \\ g + (f - g) + D_2\mathcal{F}(g, \mathrm{Id})(h_1 - \mathrm{Id}) &= g \\ \underbrace{(f - g)}_{u} + D_2\mathcal{F}(g, \mathrm{Id})\underbrace{(h_1 - \mathrm{Id})}_{w} &= 0 \end{aligned}$$

So now we may assert $w = -D_2\mathcal{F}(g, \mathrm{Id})\, u$. We hope that the sequence $u_1, u_2, \ldots$ will be decreasing as orders of each other. This is the global paradigm. Using this approach often helps when there are no other options, but it is computationally involved.

20.5 Proof Sketch for the Poincaré case

Assume $|\lambda| \neq 1$. We begin with the functions

$$f(z) = \lambda z + \sum_{i=2}^{\infty} f_i z^i, \quad \Lambda(z) = \lambda z$$

Set $u = f - \Lambda$. It may be shown that the derivative is arbitrarily small, $|u'| < \varepsilon$, in some disk B_ρ around zero. Call

$$a_k = \frac{f_k}{\lambda - \lambda^k},\ k \geqslant 2$$

This coefficient is found by solving cohomological first-order equation (we omit explanation). We may write

$$w = \sum_{k=2}^{\infty} \frac{f_k}{\lambda - \lambda^k} z^k = \sum_{k=2}^{\infty} a_k z^k, \quad h_1 = \mathrm{Id} + w$$

We know that our f approximates Λ up to the second term. We want to be sure that after we apply h_1, the new map approximates our function up to the fourth term (i.e. first three terms vanish). Recall that the conjugacy equation we are concerned with is $h \circ f = \Lambda \circ h$. If we substitute h_1 into this equation above, we get the expression $(\mathrm{Id} + w)f - \lambda(\mathrm{Id} + w)$. We would like to compute $[(\mathrm{Id} + w)f - \lambda(\mathrm{Id} + w)]z$ as this measures the failure of h to be a conjugacy. To reiterate what we have at this point

$$\begin{aligned} u(z) &= f_2 z^2 + f_3 z^3 + O(z^4) \\ w(z) &= a_2 z^2 + a_3 z^3 + O(z^4) \\ f(z) &= \lambda z + f_2 z^2 + f_3 z^3 + O(z^4) \end{aligned}$$

With these, we compute the products

$$\begin{aligned} w(f(z)) &= a_2 (f(z))^2 + a_3 (f(z))^3 + O\left(z^4\right) \\ &= a_2 \left[(\lambda z)^2 + 2\lambda f_2 z^3\right] + a_3 (\lambda z)^3 + O\left(z^4\right) \\ &= a_2 \lambda^2 z^2 + 2\lambda f_2 a_2 z^3 + a_3 \lambda^3 z^3 + O\left(z^4\right) \\ -\lambda w(z) &= -\lambda a_2 z^2 - \lambda a_3 z^3 + O(z^4) \end{aligned}$$

Expand the expression

$$\begin{aligned} [(\mathrm{Id} + w)f - \lambda(\mathrm{Id} + w)](z) &= [f + wf - \lambda - \lambda w](z) \\ &= [u - wf - \lambda w](z) \end{aligned}$$

Now, we collect second-order terms using the expressions above to see that

$$(f_2 + a_2 \lambda^2 - \lambda a_2) z = f_2 - a_2(\lambda - \lambda^2) = 0$$

For third-order terms, we have a similar cancellation and are left with the term $2\lambda f_2 a_2$, which is $O(\|u\|^2)$ so it is quadratically smaller than the original error u.

Moving on, the idea is that if $|u'| < \varepsilon$, then on some smaller disk $B_\rho(1-\Delta)$, we have $|u_1'| < \varepsilon^2 C$, and $|u_2'| < \varepsilon^4 \tilde{C}$, where the constant depends on (Δ, c, d).
We give a brief remark on the relevance of Diophantine numbers to this approximation. In the hyperbolic case, $\lambda^k - \lambda$ is bounded from zero by some constant. In the non-hyperbolic case, we have

$$|\lambda^k - \lambda| = |e^{2\pi i k\alpha} - e^{2\pi i\alpha}| = |e^{2\pi i\alpha(k-1)} - 1|$$

By some classical approximation, we may approximate the right-hand side by $2\pi\alpha(k-1) \pmod{2\pi}$ and we have the estimate on the denominator

$$|\alpha(k-1) - p| > c(k-1)^d$$

Because our k is growing, our coefficients $\frac{f^k}{c(k-1)^d}$ are small enough. Come back and elaborate on this later. This involves the problem of small divisors. We omit the complete details, but we note that we need to go through more computations and estimates to precisely show convergence, and this involves noting $u_n < \varepsilon^{2^n}$ and showing that $|(1+u_1)(1+u_2)\dots|$ is a homeomorphism.

20.6 Classification of Fatou Set Components for Rational Maps

It is possible to apply this technique in a similar setting. Start with $f : S^1 \to S^1$, let $f \in C^\omega(S^1)$ be analytic and let the rotation number $\rho(f)$ be Diophantine. By Denjoy's Theorem, we know that $f \sim R_\rho(f)$. It turns out by Arnold's theorem, we know that f and $R_\rho(f)$ are analytically conjugated. It turns out that f may be extended to some neighborhood of the circle (i.e. some annulus in the plane) and we obtain standard map on the annulus. This standard map will be uniform rotation by some angle, and this will be holomorphic conjugacy. It is possible, however, to obtain a rotation number where there is not analytic conjugacy. In the context of rational maps, these annuli are known as Arnold-Hermann rings.

We give a brief description of the classification Fatou set components.

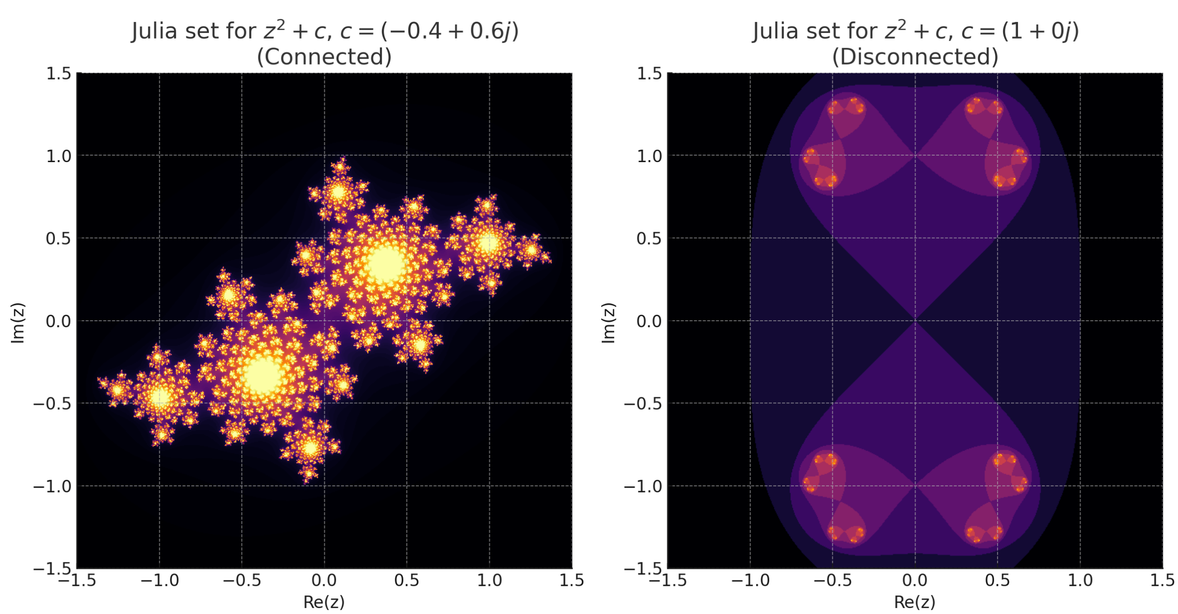
1. It may be a disk of Poincaré type. There is an attracting hyperbolic (usually fixed) point, and all trajectories within the disk move towards it. We covered this situation with the map $z \mapsto z^2$
2. We have $z \mapsto z^k$ which looks the same as the first case, but with superexponential convergence.
3. Diophantine numbers. This is called the Siegel case. We showed that this map is locally conjugated to $z \mapsto e^{2\pi i\alpha}z$. The map looks like a disk full of invariant circles. It corresponds to the 2-dimensional case of a center.

4. Arnold-Hermann ring. Ask about Blaschke product.
5. We have a flower. This case occurs with the map $z \mapsto e^{2\pi \frac{p}{q}} z + \ldots$ Here, we have q elliptic sectors. Trajectories within the ellipses go towards the center, and trajectories go around the 'perimeter' on the outside. The trajectories are structurally stable and so they belong to the Fatou set.

This situation corresponds to the case of real dynamics. We have places where dynamics are regular and places where dynamics are chaotic. We have the above situation for regular dynamics. When dynamics are chaotic, we usually rely on measure.

Once we have classified all the Fatou components (attracting basins, super-attracting basins, Siegel discs, Herman rings, and parabolic "flowers"), the complementary Julia set captures the chaotic part of the rational map.

1. The quadratic family $f_c(z) = z^2 + c$. Ask about limit explanation later again (1:26:00).
   - **Connected case.** (c in the Mandelbrot set): J_c is a connected fractal. Within a hyperbolic component of parameter space, the Hausdorff dimension $\dim_H(J_c)$ varies real-analytically with c. This follows by expressing J_c via the inverse Böttcher coordinate—which is a Hölder map on external angles—and then applying Falconer's lemma on the dimension of Hölder-continuous iterated-function systems.
   - **Disconnected case.** (c outside the Mandelbrot set): J_c is a Cantor set (a "Cantor repeller"), and f_c is an Axiom A map. Its restriction to J_c is topologically conjugate to the full one-sided shift on two symbols.



2. General hyperbolic rational maps of degree d.
   - If the Julia set is disconnected, it again breaks into uncountably many repelling cycles forming a Cantor set, with dynamics conjugate to the shift on d symbols.
   - If it is connected, one still uses the same thermodynamic-formalism tools (pressure, transfer operators, Hölder potentials) to compute dimensions of $J(f)$ and of invariant measures of maximal entropy.
3. For higher-degree polynomials or more general rational maps, the story is more complicated. Fatou components can interlace Siegel discs, Herman rings, and attracting basins, while the Julia set's geometry and dimension are studied via thermodynamic formalism (Bowen's formula, pressure functions) and symbolic models whenever the map is Axiom A.

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21.1 Flows on Tori

This lecture will cover material from Katok and Hasselblatt, P.457 onward. We previously discussed flows on the sphere, $\mathbb{S}^2$, for which we had the Andronov-Pontryagin theorem for characterization of such flows (via vector fields). As a quick review, the conditions for AP are

1. No saddle connection.
2. All fixed points are hyperbolic (i.e. saddles, sinks, or sources).
3. All periodic trajectories are limit cycles or hyperbolic.

The class of vector fields which satisfy these three conditions have two properties

1. The vector field is structurally stable.
2. The set of such vector fields is a dense and open set in the set of all vector fields of $\mathbb{S}^2$.

Now, what about other surfaces or manifolds? For orientable manifolds, the general statement was proven by Peixoto. We prefer to investigate the case of the torus because it is representative; all other manifolds are very similar.

21.2 Fixed-point Free Flow on a Torus

In this section, we are interested in flows on the torus with no fixed-points. The torus is the only orientable surface which allows such flows to exist. This is a consequence of the index theorem. In contrast, there are no vector fields on the sphere with no singular (fixed) points. Previously, we have discussed a *linear flow* on a torus. This is an example of a flow on the torus with no fixed-points.

$$\varphi_t^\omega(x) = x + \omega t$$

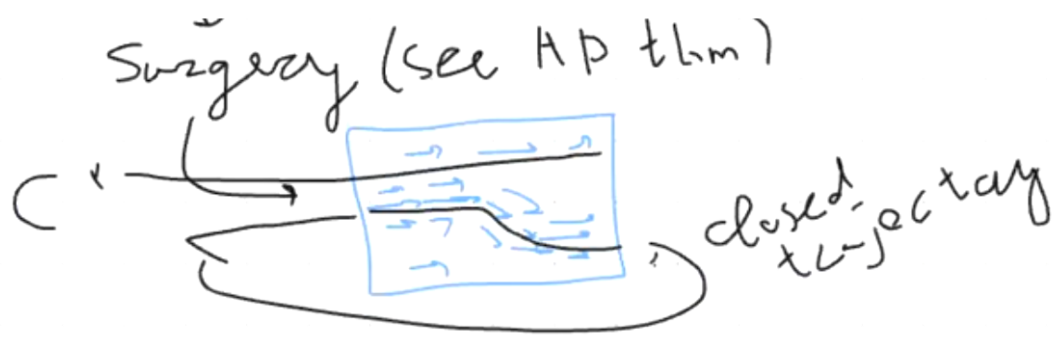
This flow acts as a parallel shift. We want to draw a comparison between all other flows with no fixed points with these linear flows. Recall the idea of a *transversal*. It is some curve on the manifold for which every tangent vector of the curve is not parallel to the vector field associated with the flow at that point.

Proposition. A fixed-point free, $\mathscr{C}^1$ flow on $\mathbb{T}^2$ admits a closed transversal (i.e. in the sense of a curve).

Remark. The point of the proposition is to be able to use the Poincaré map and go from our flow on the torus to a diffeomorphism on the circle, a concept which we have discussed in past lectures (1) (2).

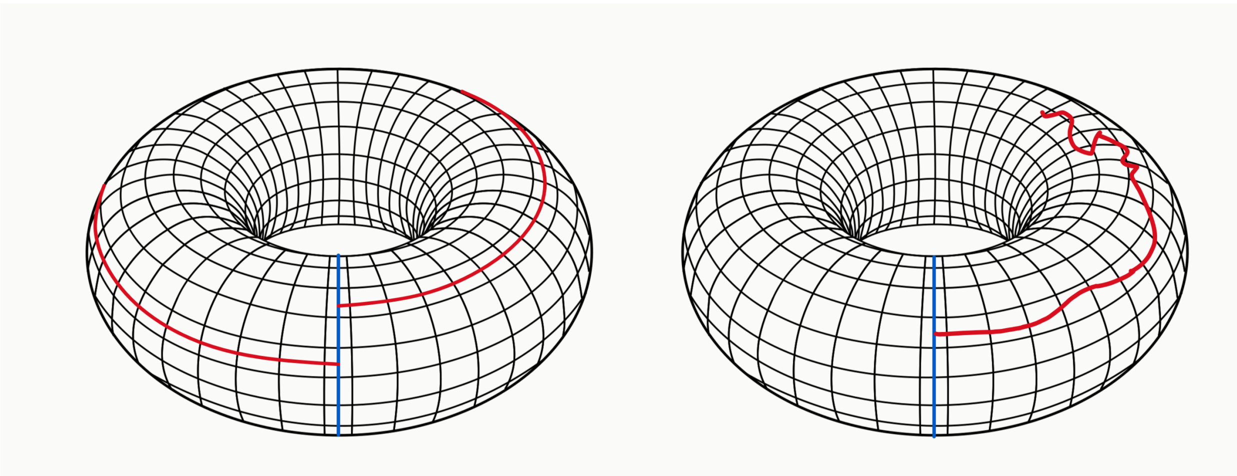
Proof. We may define a vector field X from our flow. Now, at any other point on the torus, we may define another vector field $Z = R_{\frac{\pi}{2}} X$ by rotation, which is orthogonal to X. Suppose that Z has a closed trajectory (limit cycle), then we are done. By definition, Z will not share any tangent vectors with X, so such a closed trajectory is indeed a closed transversal. Now, suppose that no limit cycle exists in Z. We consider a point x on the torus and its ω-limit set under Z, denoted $\omega_Z(x)$. Since we are on a torus, we do not have the convenient characterization of such ω-limit sets by Poincaré-Bendixson. Regardless, we may choose a point $p \in \omega_Z(x)$ and by the premise, such a point is not a fixed point. Recall the theorem from ODE on the continuity of dependence from initial conditions. Hence, near our point p, we may represent our flow as parallel flow (in some change of coordinates).

As shown in the picture above, since x and p do not belong to a limit cycle, there cannot be a finite number of times which the flow from x intersects the neighborhood of parallel flows around p. So, there are infinitely many returns of this trajectory to the neighborhood indicated by the blue box. Since there are infinitely many returns, the distances between the trajectories may be made arbitrarily small. Similar to our discussion in the Andronov-Pontryagin theorem, there is the concept of surgery: we may change our vector field by a $\mathscr{C}^1$ perturbation such that the disjoint trajectories which are ε far apart now coincide with each other after the perturbation.



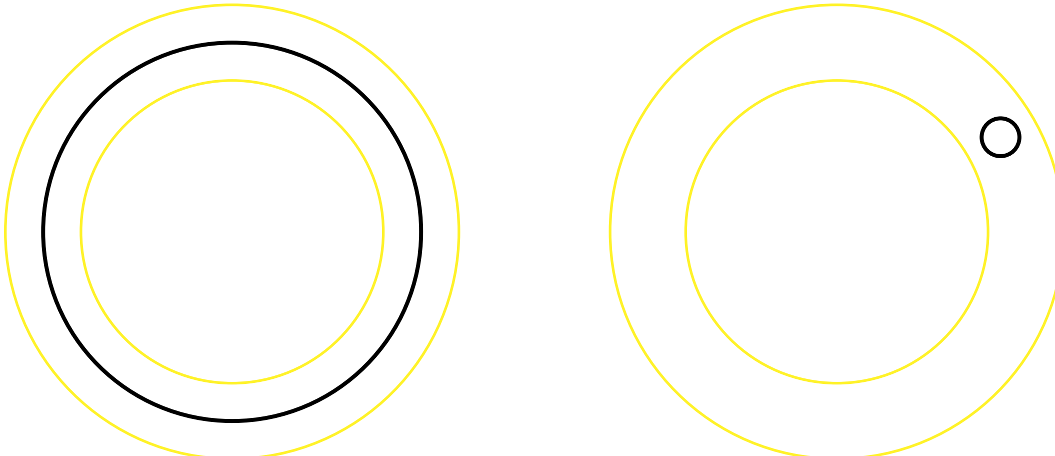
From this perturbation, we obtain a new flow, say $\overline{Z}$, which differs negligibly from Z such that the vectors of $\overline{Z}$ along the trajectory are still not parallel to the tangent vectors of the curve. Outside the area of the blue box, we will still have orthogonality, but within the blue box, the perturbation may result in vectors that are no longer orthogonal to the tangent vectors of the curve, but not parallel to the tangent vectors either. Hence, such a closed trajectory of $\overline{Z}$ will be our desired transversal. ■

We now turn our attention to the **Poincaré map**. Given a transversal, there are two possible situations. Given some point on the transversal, it is possible that the point returns to the transversal, or it does not return, under the flow.



Proposition. The Poincaré map is either not defined at any point on the torus, or well-defined (i.e. exists at every point).

Proof. By the previous proposition, there exists a (nontrivial) closed transversal on the torus. We may cut our torus with respect to this transversal. The resulting surface is a cylinder. Moreover, the cylinder may be represented as an annulus (if we imagine widening one of the openings of the cylinder and projecting it onto the plane). The flow on this annulus will appear as lines that travel across, which teleport when hitting the inner or outer circles of the annulus, depending on the identification. If the Poincaré map is well-defined everywhere (i.e. all points return to the transversal), we are done. So, suppose there is a trajectory of the point x which does not return to the transversal. From the point of view of this trajectory, there is no such 'teleportation' associated with identified points; it is just a trajectory on an annulus. A trajectory on an annulus is a trajectory on a sphere. Thus, we may invoke the Poincaré-Bendixson theorem on the annulus to conclude that $\omega(x)$ must be a fixed point, cycle, or polycycle. Since we are on a fixed-point free flow, $\omega(x)$ cannot be a fixed point nor a polycycle. We see that $\omega(x)$ must be a cycle, which we call Γ. There are two options.



As in the picture above, either the cycle contains the circle in the center, or it does not. In the first case, Γ separates the boundaries: there are no trajectories which may go from one boundary to the other. In this case, the Poincaré map is not defined. This situation is similar to flows on the sphere. Now, in the second case, we see that Γ is trivial. So, we have a flow inside this cycle which is tangential to it. From topology, Brouwer's theorem which guarantees the existence of a singular point. Hence, this case is impossible because it contradicts our premise of a fixed-point free flow.

Lastly, consider the case where our transversal is trivial on the torus. This case is also impossible for a similar reason. Recall that in the previous proposition, we constructed a vector field Z or $\overline{Z}$ to obtain our transversal. If either of these flows is trivial, then we may invoke Brouwer's theorem again to obtain a fixed point, contradicting our premise. ■

21.3 Poincaré Map on Torus

Above, we have delineated two situations for the Poincaré map on a torus. If it is not defined at all, then the dynamics resemble that of a sphere. Our discussion now focuses on the case where the Poincaré map is well-defined.

Proposition. Every fixed-point free $\mathscr{C}^1$-flow with no cycles on $\mathbb{T}^2$ is orbitally conjugated to Smale construction (suspension) over a circle map.

Remark. Recall that when we are working with diffeomorphisms, there is only topological conjugacy. However, with flows, there is both topological and orbital topological conjugacy (change time). Also, we must specify that there are no cycles to exclude spherical dynamics.

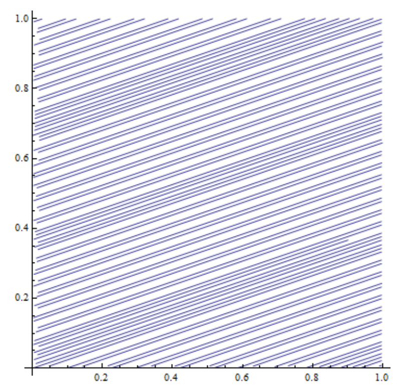
Proof Sketch. We begin with a nontrivial transversal for the flow. Call the flow φ and the transversal Σ. Note that Σ is diffeomorphic to the circle. Since we have excluded cycles, by the previous proposition, we must have a well-defined Poincaré map P. So, given a point x on the transversal, it returns at $P(x)$. What is the time of this return? Let us denote, for any point x, the angular coordinate θ on the circle. To be explicit, choose a base point on Σ, record the arclength, and normalize that length to 2π to give the smooth coordinate

$$\theta : \Sigma \to S^1 = {}^{\mathbb{R}}/_{2\pi\mathbb{Z}}$$

The point of θ is to tell us where along the transversal we are. Let $h(x)$ be the time of first return. We introduce new coordinates on the torus, (θ, y), where $0 \leqslant y < h(\theta)$. Thus, in this torus, we have a standard flow $\frac{\partial}{\partial y}$ and circle map $\theta \mapsto P(\theta)$. So this is just a vertical flow. Now, in principle, our flow φ may go up at different speeds for different points along the transversal. We want to change time such that this upward speed will be the uniform. After this change, this modified flow φ will be orbitally conjugated to the Smale suspension over circle map. To summarize our idea:

1. Given the transversal and the premises above, there is a first return map.
2. There are trajectories from each point x on the transversal that define $P(x)$, another point on the circle and we have a map from the circle to itself.
3. Now we take time as a parameter through which we move (on the torus) from our initial point to the point of first return. So time is a new coordinate on our torus.
4. After this change of coordinates, the map becomes the Smale construction.

If the rotation number $\rho(P)$ of our Poincaré map is irrational, we may again change coordinates via $\theta \mapsto \theta + \rho$. In this case, we may say that the flow φ is conjugated up to changes in coordinates in time to Smale construction over rotation. Note that if we glue the boundaries by this rule $(x \mapsto x + \rho)$, this vertical flow will become slightly angled (diagonal direction). So, in standard coordinates, Smale construction over rotation looks like a *linear flow*.



For irrational rotations, we have this conjugacy to linear flow.

Aside. We have previously discussed different types of conjugacy and rotation numbers. For instance, if our rotation number is Diophantine, then this conjugacy may be done smoothly.

21.4 Flows on Torus with Fixed Points

The most prominent object for these flows is the **Cherry cell**. We have a fixed point which is a saddle and which connects to three other fixed points via separatrices. Additionally, there is a sink in of the sections, below and to the left of the saddle.

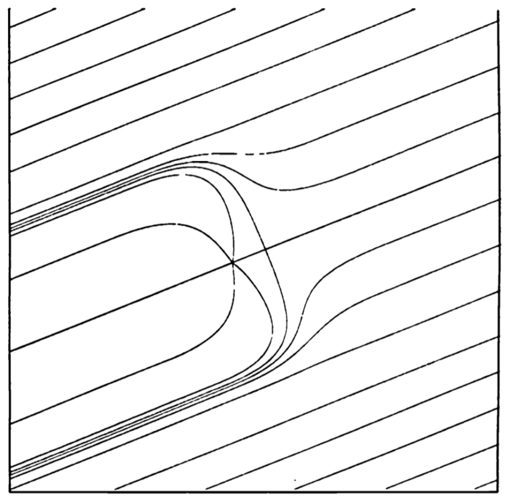
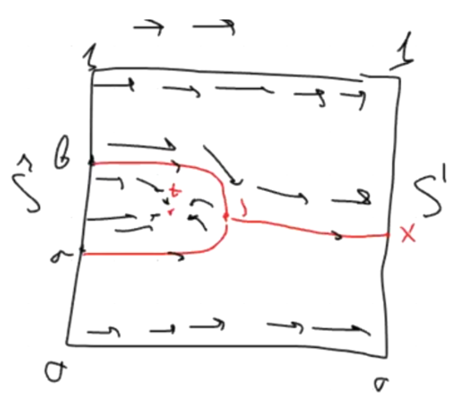


FIGURE 14.4.1. The Cherry flow

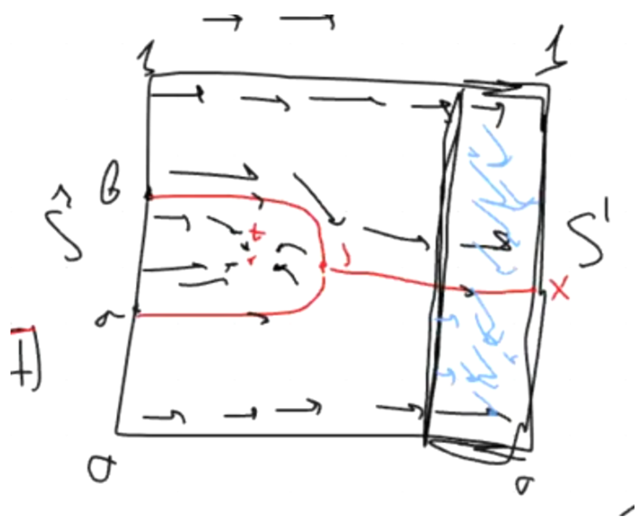
Why are we concerned with such an object? The idea is we can take a flow where everything proceeds in the same direction (i.e. nothing is happening) and replace it with a Cherry cell. In particular, we may replace a linear flow with a Cherry cell. We end up with a picture as below



We see that for $S^1\backslash[a,b]$ the Poincaré map is well-defined and maps into $S^1\backslash\{x\}$. We may redefine our Poincaré map on $[a,b]$ by taking $P([a,b]) = x$. By doing so, the map will no longer be a diffeomorphism and not strictly monotonic. However, notice that our facts about rotation number do not require the function to be a diffeomorphism nor strictly monotonic, we just need a monotone function. So, we have the properties we have proved before

1. $\rho(P) \in \mathbb{Q}$ if and only if there is a periodic orbit.
2. If $\rho(P) \notin \mathbb{Q}$, then $\omega(y) = S^1$ or $\omega(y) = K$ (where K is some Cantor set).

Note that there is no Denjoy theorem because of the lack of an inverse. There is another modification of concern, where we take a section of the torus and shift the field downward, as indicated below by the blue arrows.



Such a change may be made smoothly. So, we may turn it by a small number α. It may be seen that $\rho(P)$ continuously changes, and we may pick $\rho(P) \notin \mathbb{Q}$. We are interested in this case where our rotation number is irrational. In this case, we may be sure that $x \notin [a, b]$, otherwise, our flow under the first return map returns to x, that is $\varphi_t(x) = x$ and we have a periodic orbit. By the same argument, we have $f^n(x) \notin [a, b]$. Hence, we have $\omega(x) \cap (a, b) = \varnothing$. But, by our Poincaré classification theorem, there are only two options: either $\omega(x) = S^1$ or $\omega(x) = K$. Since $\omega(x) \neq S^1$, we must have $\omega(x) = K$. Here, we have shown that the ω-limit set of any point x must be a Cantor set.
We have investigated ω-limit sets of points outside of $[a, b]$, what about points in $[a, b]$? It turns out that some ω-limit sets in this interval do not correspond to real trajectories due to the presence of a sink. For instance, call this sink a and let $p \in [a, b]$. Then, we see that $\omega(p) = a$, but that we had defined $P(p) = x$. Thus, if we choose a point that hits the sink a, our ω-limit set of our flow will not correspond to the ω-limit set of the Poincaré map. However, in all other cases there is a correspondence, that is

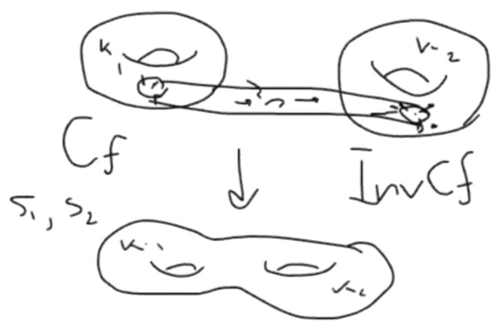
$$\omega_\varphi(y) \cap S^1 = K$$

where S^1 is one of the vertical lines in our picture. So, our ω-limit set of the flow looks like a union of Cantor sets which smoothly depends on the horizontal axis. Additionally, all points in this set are not singular. This suggests a different type of ω-limit set.

By Poincaré-Bendixson, on the sphere S^2, our ω-limit sets may be a fixed point, cycle, or polycycle. In contrast, on tori and all other surfaces, there is another type of ω-limit set called the **nontrivial recurrence.** This ω-limit set is not a cycle, polycycle, or fixed point. In fact, there are no fixed points in this type of set. This nontrivial recurrence is the reason why it is more difficult to classify vector fields on a torus in comparison to vector fields on a sphere.

21.5 More General Characterizations of Flows

Suppose we have a Cherry flow on a torus and the inverse Cherry flow on another torus. The Cherry flow has an attracting point while the inverse Cherry flow has a repelling point. We may make circular cuts in the regions around these points and connect them together by a tube to obtain a pretzel and a vector field with two saddles on that pretzel (with no attracting or repelling points).

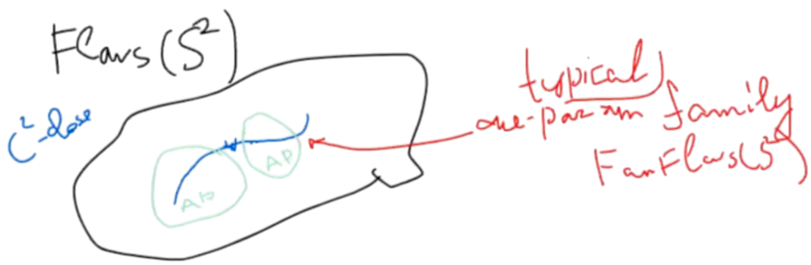


So, there are two nontrivial recurrent sets K_1, K_2 which live in this tori. Further, any point on the pretzel is attracting to one of two Cantor sets, each which exist on different tori, so $\omega_p(x) \in K_i$ for $i = 1, 2$. This statement applies up to separatrices of a saddle. It is a bit counterintuitive since we are imagining that points on the surface are sorting themselves out into two separate sets. The fifth option is $\omega(x) = \mathbb{T}^2$, recall that this occurs in fixed-point free dynamics and this is specific to the torus. We list the other options for the ω-limit set again, which occur on any surface:

1. Fixed point
2. Cycle
3. Polycycle
4. Nontrivial Recurrence

These options characterize dynamics on orientable surfaces. Non-orientable surfaces have a special type of dynamics called flip-flop loops (i.e. on Klein bottle).

Recall now that we had the Andronov-Pontryagin criteria to characterize flows on the sphere. The generalization of the AP theorem which applies to the torus is Peixoto's theorem. However, there are other ways to generalize the results of the AP theorem. Consider the space of flows on the sphere, Flows($\mathbb{S}^2$). Suppose that there is a curve in this space such that any two maps on this curve are $\mathscr{C}^2$-close to each other. This curve is called the **one-parametric family**. Now, most points on our curve belong to a dense and open set where the Andronov-Pontryagin theorem applies. However, it is possible to move our curves outside of these regions. We exclude these cases by restricting our investigation to **typical** one-parametric families, call this FamFlow($\mathbb{S}^2$).



Our following discussion will focus on more intuitive explanations. Typical one-parametric family fall into a few situations. One possibility is that the entire family is an Andronov-Pontryagin set, i.e. nothing happens. The other possibilities, which involve degeneracies, are described by Jorge Sotomayor.

1. One field has a saddle connection.
2. The fields have a saddle loop.
3. A parabolic cycle exists for one map in the family.
4. Andronov-Hopf Bifurcation. Qualitatively, focus into cycle.
5. Saddle node, which looks like a saddle from one side, and node from the other side.
6. Saddle node loop.

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22.1 Flows on Multidimensional Tori

Recall that we may go from a circle rotation $\varphi \mapsto \varphi + \alpha$ to a linear flow on a torus, $\mathbb{T}^2$. Likewise, if we have a linear flow on an n-dimensional torus $\vec{x} \mapsto \vec{x} + \vec{\gamma}$, we obtain a linear flow on an $(n+1)$-dimensional torus, $g^t(x) = x + \Omega t$.

> *Aside.* Usually there is a many-to-many relation between the objects mapped by the Poincaré and Smale maps. However, if one fixes a particular transversal, we can make this into a one-to-one correspondence. We can change transversals via a Möbius transformation.

Suppose we start with a map T, a shift, given by

$$T(x_1, \ldots, x_n) = (x_1 + \gamma_1, \ldots, x_n + \gamma_n)$$

When we spoke about rotation, there were two cases: rational rotation and irrational rotation. So, the dynamics were either totally regular or totally chaotic; the space was split up into all dense periodic orbits, or no periodic orbits at all. Recall our $\mathscr{C}^1$-structural stability theorem for circle diffeomorphisms as a related discussion. For higher-dimensional cases, we may have dynamics which appear to resemble something in between. For instance, suppose $\vec{\gamma} = (\gamma_1, 0)$ where $\gamma_1 \notin \mathbb{Q}$. Then, for our trajectories, all points move along vertical circles and in any circle, we have chaotic dynamics, but the dynamics are not chaotic on the entire torus.

As a starting point, we are interested in the most chaotic case.

Definition. *Rationally independent*
The real numbers $\gamma_1, \ldots, \gamma_n$ are rationally independent if

$$\sum_{k=1}^{n} k_i \gamma_i = 0 \implies k_1 = \ldots = k_n = 0$$

where $k_1, \ldots, k_n \in \mathbb{Z}$.

Proposition. All trajectories of linear shift on $\mathbb{T}^n$ are dense if and only if all of our shifts $\gamma_1, \ldots, \gamma_n$ satisfy

$$\sum_{k=1}^{n} k_i \gamma_i \in \mathbb{Z} \implies k_1 = \ldots = k_n = 0$$

for $k_1, \ldots, k_n \in \mathbb{Z}$.

Proof. Assume $\sum_{k=1}^{n} k_i\gamma_i = k$ for nontrivial coefficients $k_1, \dots, k_n$. Define a function $\varphi : \mathbb{T}^n \to \mathbb{R}$ by

$$\varphi(x) = \sin\left(2\pi \sum_i k_i x_i\right)$$

We claim that this function is invariant. Observe that

$$\varphi(x+\gamma) = \sin\left(2\pi \sum_i k_i x_i + 2\pi \underbrace{\sum_i k_i \gamma_i}_{k}\right) = \varphi(x)$$

So, by this invariance, we see that

$$\varphi(\mathrm{Orb}(x)) = \mathrm{const}(x)$$

The function φ is not constant because the sum $\sum_i k_i x_i$ is not trivial. Choose a value a of φ which is not the maximum nor minimum. Define the sets

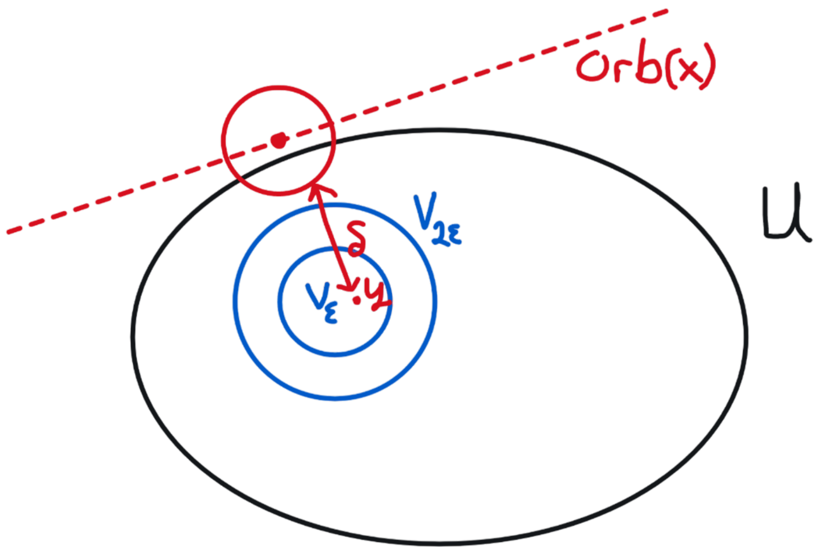
$$A = \{x : \varphi(x) < a\}, \quad B = \{x : \varphi(x) > a\}$$

Note that these are preimages. Both of these sets are open because our function is continuous and they are nonempty because we chose a as a value that is neither the maximum nor minimum. Thus, on our multidimensional torus, we have the two sets A and B. Further, by the invariance of the orbits, we see that if $x \in A$, then $\varphi^2(x) = \varphi(x) < a$ so that $\varphi(x) \in A$ as well. Inductively, $\varphi^n(x) \in A$ for all $n \in \mathbb{N}$. It follows that if $x \in A$, then $\mathrm{Orb}(x) \subseteq A$ and a similar statement applies for $x \in B$. Therefore, the orbits are not dense, since they either never visit A or never visit B.

On the other hand, suppose that our orbits are not dense. Let $x, y \in \mathbb{T}^n$ be two distinct points. Since $x + n\gamma = y + n\gamma + (x - y)$, we see that $\mathrm{Orb}(x) = \mathrm{Orb}(y) + (x - y)$. Since our orbit is not dense, the set $U = \mathbb{T}^n \setminus \overline{\mathrm{Orb}(x)}$ is open and nonempty. Take a small neighborhood around x, call it $U_\delta(x)$. Consider the set

$$\mathrm{Orb}(U_\delta(x)) = \bigcup_{y \in U_\delta(x))} \mathrm{Orb}(y)$$

Now, take the set $U_1 = \mathbb{T}^n \setminus \overline{\mathrm{Orb}(U_\delta(x))}$. We first show U_1 is nonempty. Since we know that U is nonempty, choose an $\varepsilon > \delta > 0$ and consider two concentric epsilon balls of V_ε, and $V_{2\varepsilon}$ contained in U. Any trajectory in $\mathrm{Orb}(U_\delta(x))$ should not pass through our ball of radius 2ε. Suppose, for contradiction, that $y \in \mathrm{Orb}(U_\delta(x)) \cap V_\varepsilon$. Since $V_{2\varepsilon} \subseteq U$, no orbit of x belongs to $V_{2\varepsilon}$. Since $y \in V_\varepsilon$ it is at least ε away from any orbit of $\mathrm{Orb}(U_\delta(x))$, but this contradicts its membership in $\mathrm{Orb}(U_\delta(x))$ since $\varepsilon > \delta$. Thus U_1 is nonempty.



Above is a diagram of the contradictory situation. Continuing, let us denote $V_1 = \mathrm{Orb}(U_\delta(x))$. Notice that V_1 and U_1 are two invariant open sets. U_1 is invariant because the complement preserves invariance. Consider the characteristic function

$$\chi_{U_1}(x) = \begin{cases} 1 & x \in U_1 \\ 0 & x \notin U_1 \end{cases}$$

Since this is a measurable function, it has a Fourier representation. We write

$$\chi_{U_1}(x_1, \ldots, x_n) = \sum_{k_1, \ldots, k_n \in \mathbb{Z}_n} x_{1 \ldots n} e^{2\pi i \left(\sum_{j=1}^n k_j x_j\right)}$$

Since U_1 is invariant under the shift transformation, we see that $\chi(x + \gamma) = \chi(x)$. We have a corresponding representation of the shifted function

$$\chi_{U_1}(x_1 + \gamma_1, \ldots, x_n + \gamma_n) = \sum x_{1 \ldots n} e^{2\pi i \left(\sum_{j=1}^n k_j \gamma_j\right)} e^{2\pi i \left(\sum_{j=1}^n k_j x_j\right)}$$

Since these expressions are equal, their coefficients must be identical. Hence, we see that

$$x_{1 \ldots n} = x_{1 \ldots n} e^{2\pi i \left(\sum_{j=1}^n k_j \gamma_j\right)}$$

Suppose, for contradiction, that $\sum_{k=1}^n k_i \gamma_i \in \mathbb{Z}$ implies that $k_1, \ldots, k_n$ are all 0. So, if $k_1, \ldots, k_n$ are not all zero, then $x_{1 \ldots n} = 0$. Hence, we see that

$$\chi_{U_1} = x_{0 \ldots 0} \cdot 1 = C$$

where C is some constant and where this equality holds almost everywhere. But V_1 is a nonempty open set, so that it has positive measure, so there must be some $y \in V_1$ so that $\chi_{U_1}(y) = 1$ ($\lightning$).

■

22.2 Linear Flows in Mechanics

V.I. Arnold's book *Mathematical Methods of Classical Mechanics* is especially germane and recommended for this discussion. We now discuss manifolds, vector fields on manifolds, and differential forms on manifolds. Let (M^{2n}, ω^2) be a pair consisting of a manifold with even dimensional and a closed, nondegenerate 2-form. This defines a structure called a *symplectic manifold*. This is an alternative structure to the Riemannian manifold. For a Riemannian structure on a manifold, locally, things may look different (i.e. compare a part of a sphere, a plane, and part of the Lobachevsky plane, these differ due to their Gaussian curvature). The symplectic structure is skew-symmetric, unlike the Riemannian structure, and it always looks the same (canonical Darboux coordinates).
This form defines a canonical isomorphism I. It is easier to start from the inverse object. Let $\xi \in TM^{2n}$ be vector field on the tangent space. We can send this vector field to 1-forms by

$$I^{-1} : \xi \to \omega(\xi, \cdot)$$

So, it acts from the tangent space TM to the cotangent space T^*M. Suppose we have a function that is denoted $H : M \to \mathbb{R}$, since it will be Hamiltonian. Recall that when we defined flows on surfaces such as a sphere, one way to think about flows are as arising from a vector field v, when we consider solutions to the ODE $\dot{x} = v(x)$. Similarly, we would like to define a flow on our manifold by

$$\dot{x} = I \cdot dH(x)$$

As a sanity check, $dH \in T^*M$ and the canonical isomorphism I sends it to the tangent space TM. So, it is a vector field. The above is a flow which is called a **Hamiltonian flow.**

Theorem. *Liouville Theorem*
It is also called Arnold's Theorem and in higher dimensions it is called the Liouville-Arnold-Jost Theorem.
Let M be a symplectic manifold. Let $F_1, \ldots, F_n$ be functions in involution. We have n flows defined by the equations

$$\vec{x} = I\, F_i(x)$$

Call each flow resulting from these vector fields g_i. Being **in involution** means that these flows commute

$$g_i^t \circ g_j^s = g_j^s \circ g_i^t$$

Choose one function to denote $H = F_s$. Define the sets

$$M_c = \{F_i(x) = c_i\}$$

We also require that our differentials dF_i are linearly independent on M_c. Given these conditions, we have:

1. M_c is a smooth, g_i-invariant manifold.
2. If M_c is compact and connected, then $\mathbb{T}^n \sim M_c$ (i.e. a diffeomorphism exists)
3. Flow g_H on M_c is linear:
$$g_H^t(\vec{x}) = \vec{x} + \vec{\Omega}_c t$$

Remark. In practice, these n functions in involution are n invariants of your system, say angular momentum, energy, and so forth. So, all integrable systems have this type of dynamics in canonical coordinates and they may be integrated! Unfortunately, many systems are not integrable.

Suppose we have functions that are in involution, but only under perturbation. Then, the toral structure persists: it appears as a tori that fill the space in addition to some axis in the middle which is not a closed manifold. Some manifolds are unions of the tori which are not connected. It was proved that under perturbation, most "Diophantine" (sufficiently irrational–frequency) tori persist, while resonant ones break into complicated Cantor-like remains ("cantori") This is related to KAM theory.

Another question is what happens on the torus itself? Suppose that the entries of $\vec{\Omega}_c$ are rationally independent. Then something similar to the Kronecker-Weyl theorem happens, there are some ergodic properties (i.e. we could describe average properties). If not, there are problems, and these are called resonances.

Proof sketch. We know that M_c is a smooth manifold by the implicit function theorem. From a proposition in differential geometry, the condition $g_i^t \circ g_j^s = g_j^s \circ g_i^t$ allows us to conclude that $F_i(g_j^t(x)) = \text{const}$. This shows why M_c is g_i-invariant.
Next, we show the second conclusion regarding its toral structure. We define a multiflow $g^{(\cdot)} : \mathbb{R}^n \to M_c$ by
$$g^{\vec{t}}(x) = g_1^{t_1} \circ \ldots \circ g_n^{t_n}(x)$$
Because our M_c is invariant, if we start from a point on this flow, we obtain a point on this manifold again. By our premises, we know that differentials are nondegenerate and linearly independent. Consider a small neighborhood around $\vec{0} \in \mathbb{R}^n$. We may map this neighborhood by $g^{(\cdot)}$ to the neighborhood of $x \in M_c$.

We claim that g^t is onto. Take two distinct points $x, y \in M_c$ and take any curve that goes from x to y. Such a curve exists because we assume that M_c is connected. Consider neighborhood U of x (let it be the image of the neighborhood around $\vec{0}$). For any point a along the curve, take the same neighborhood from $\vec{0}$ to the neighborhood of a (this is possible by translation).

In this way, we can cover the curve with images of g. Since our space is compact, there are a finite number of neighborhoods which covers the entire curve. So, we have a curve from x to y with a finite covering via neighborhoods. So, we may write g as the composition

$$g^t(x) = g^{\varepsilon_N} \circ \ldots \circ g^{\varepsilon_1}(x) = y$$

I think the above claim needs some more rigor in how it is explained.

We define the stationary group. Call the group Γ, we say that $t \in \Gamma$ if $g^t(x) = x$. It turns out this group is discrete. To see this, note that a small neighborhood around zero is mapped bijectively into a small neighborhood around x, since our differentials are nondegenerate. Thus, this group has no points near zero. In $\mathbb{R}^n$, if our subgroup has no points near zero, then we know that the group is discrete. Further, the stationary group does not depend on how we choose our point. For instance, choosing some other point y, there exists a τ such that $g^\tau(x) = y$ and applying this we have

$$g^t(y) = g^t g^\tau(x) = g^\tau g^t(x) = g^\tau(x) = y$$

Hence, we may define any point of M_c as

$$M_c \sim \mathbb{R}^n / \Gamma$$

The elements of the discrete group Γ may be described as $\Gamma = (m_1 e_1 + \ldots + m_k e_k)$, where $m_1, \ldots, m_k \in \mathbb{Z}$ and $e_1, \ldots, e_k$ are some vectors.

Consider the set $\mathbb{T}^k \times \mathbb{R}^{n-k} = (\underbrace{\varphi_1, \ldots, \varphi_k}_{\in \mathbb{T}^k}, \underbrace{y_1, \ldots, y_{n-k}}_{\in \mathbb{R}^{n-k}})$. Now, we define a map $P : \mathbb{R}^n \to \mathbb{T}^k \times \mathbb{R}^{n-k}$ by

$$(x_1, \ldots, x_k, y_1, \ldots, y_{n-k}) \mapsto (x_1 \pmod 1, \ldots, x_k \pmod 1, y_1, \ldots, y_{n-k})$$

We may assert that there exists a linear isomorphism A, which is a change of coordinates, on $\mathbb{R}^n$. We define this map by $(x, \ldots, y) \mapsto (e_1, \ldots, e_k, y_1, \ldots, y)$, where $(e_1, \ldots e_n) \in \Gamma$. We have

$$\begin{array}{ccc} \mathbb{R}^n & \xleftrightarrow{\quad A \quad} & \mathbb{R}^n \\ {\scriptstyle P}\big\downarrow & & \big\downarrow{\scriptstyle g^t} \\ \mathbb{T}^k \times \mathbb{R}^{n-k} & \xleftrightarrow{\quad \tilde{A} \quad} & M_c \end{array}$$

It turns out that for any lifting from $\mathbb{T}^k \times \mathbb{R}^{n-k}$ to $\mathbb{R}^n$ and projection from $\mathbb{R}^n$ to M_c, an automorphism $\tilde{A}$ is defined between $\mathbb{T}^k \times \mathbb{R}^{n-k}$ and M_c. $\tilde{A}$ is a diffeomorphism. If M_c is compact, then $n - k = 0$ so the above statement shows that $M_c \sim \mathbb{T}^k$.
We discuss how our flow acts on this manifold. In these coordinates, g_H corresponds to some flow $g^{t(1,0,\ldots,0)}$. In our multidimensional flow, it correspond to the case where we choose vectors only from one line. We may project this line via our diffeomorphism to a trajectory on the torus, $x \mapsto x + \Omega t$. So, once we have determined our diffeomorphism from the previous step, it not only defines the diffeomorphism between the manifold and torus, but also a canonical map where the first integral acts as a flow.

■

22.3 Applications

Suppose we have the system

$$\dot{x} = \frac{\partial H(x,y)}{\partial y}$$
$$\dot{y} = -\frac{\partial H(x,y)}{\partial x}$$

We may compute

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt}$$
$$= \frac{\partial H}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial H}{\partial y}\frac{\partial H}{\partial x} = 0$$

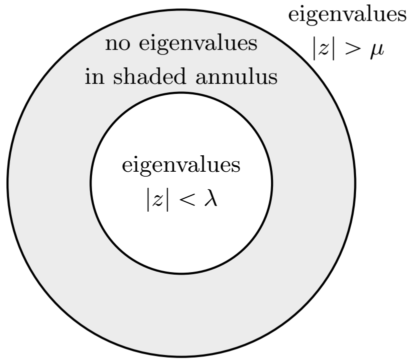
Hence, H is constant for solutions. What is the topology of this curve and what do they look like? If we try to make a picture of this, ignoring the degenerate case, we either have a one-dimensional torus or zero-dimensional torus. Check back on discussion here 1:27:00 for more detail.

23 May 14, 2025

In this lecture, our focus will be on proving the Hadamard-Perron theorem.

23.1 The Hadamard-Perron Theorem

We start with the situation where there is one hyperbolic point. This is part of the local theory of dynamical systems. The below proof will apply to periodic points, but we consider fixed points for concreteness. Let f be a diffeomorphism with a fixed, hyperbolic point O. As a reminder, hyperbolicity means $|Df(O)| \neq 1$. Thinking in the context of the complex plane, since the eigenvalues are not equal to 1, they are bounded away from 1. Hence, there exist real numbers λ, μ for which our eigenvalues exist when they are less than λ or greater than μ. In other words, in the annulus between μ and λ, there are no eigenvalues.



Recall that we may define local stable and unstable manifolds, as below.

$$W_s^{\text{loc}} = \{x : \omega(x) = O\}$$
$$W_u^{\text{loc}} = \{x : \alpha(x) = O\}$$

Global stable and unstable manifolds are defined by

$$W_s = \bigcup_{n=0}^{\infty} f^{-n}(W_s^{\text{loc}}), \quad W_u = \bigcup_{n=0}^{\infty} f^{n}(W_u^{\text{loc}})$$

Theorem. *Hadamard-Perron Theorem*
Let M be a manifold. If $f \in \mathscr{C}^r(M)$, then W_s, W_u are $\mathscr{C}^r$-manifolds.

Proof. We will only prove the case for $r = 1$. Instead of thinking of $f : M \to M$, we will think of it as $f : U \to \mathbb{R}^n$, where $U \subseteq \mathbb{R}^n$ and n is the dimension of our manifold. The main idea is that our differential, Df, acts in some basis as a block matrix (i.e. Jordan form of some matrix). We have

$$Df = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)$$

where A will correspond to those eigenvalues such that $|\mathrm{sp}(A)| > \mu$ and B will correspond to eigenvalues satisfying $|\mathrm{sp}(B)| < \lambda$. Further, we may choose corresponding coordinates x for A and y for B. So, in these coordinates, f may be represented as follows

$$f(x, y) = (Ax + \alpha(x, y), By + \beta(x, y))$$

where $(x, y) \in \mathbb{R}^n$ and α, β are some small values (i.e. we are just saying that it is close to the value of its differential).
At this point, we discuss the structure and plan of the proof.

1. Find an invariant distribution, which are fields of linear spaces. For instance, let the dimension of the x coordinates be k and the dimension of the y coordinates be $n - k$. We want to obtain k dimensional and $n - k$ dimensional distributions, E^+ and E^-, which are invariant under the differential.

$$f_x(E^+) = E^+$$

   We remark that invariant distributions are very useful objects due to their important to the theory of integrability of ODEs and PDEs.

2. Find invariant graphs of Lipschitz functions. To be concrete, suppose we have a function $f : \mathbb{R}^2 \to \mathbb{R}^2$. If we have a graph in $\mathbb{R}^2$, given by $(x, \varphi(x))$, then our function f acts on the graph through

$$(x, \varphi(x)) \mapsto f(x, \varphi(x))$$

   We are interested in certain invariant graphs with properties of invariant manifolds. Essentially, the same direction (from conic structure) and dimensions. The point is invariant graphs correspond to invariant manifolds.

3. At this point, we have an invariant distribution and invariant graph, and we want to connect these two objects. We will prove that invariant distributions are to these invariant Lipschitz functions. This means that we will have a tangent space at every point so that our functions are smooth and our manifolds are $\mathscr{C}^1$-smooth.

Remark. This proof may be modified slightly at every step to guarantee that these manifolds, which will not be invariant but will be sent one to another, exist for any point in a small neighborhood of our zero point. So, while the manifolds individually may not be invariant, but the family of the manifolds is invariant.

We now begin the steps of our proof. See also P.237-260 of Katok and Hasselblatt. We split

$$\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n-k}$$

We will take letters $u \in \mathbb{R}^k$ and $v \in \mathbb{R}^{n-k}$, so that (u, v) is an element of this direct sum. We now define horizontal and vertical cones.

$$H_p^\gamma = \{(u, v) \in T_p\mathbb{R}^n, \|v\| \leqslant \gamma \|u\|\}$$
$$V_p^\gamma = \{(u, v) \in T_p\mathbb{R}^n, \|u\| \leqslant \gamma \|v\|\}$$

Note that γ is some constant which corresponds to the angle into the vertex of the cone.

Aside. It is possible for a space to be split into two cones (simply imagine a cone and its complement set in $\mathbb{R}^2$ or $\mathbb{R}^3$), but in our case it is not necessary to require this for the tangent space.

Now, we define these cones for every point in the manifold, so, these structures are defined on the tangent bundle. We may act on the tangent bundle by the action

$$(f_* K)_p = Df_{f^{-1}(p)}(K_{f^{-1}(p)})$$

Ask for a picture of this notation. On an intuitive level, over iterations by this action, we are getting thinner and thinner cones, the limit of which is a line. This limit will be our invariant distribution. We return to our representation of f.

$$f(x, y) = (Ax + \alpha(x, y), By + \beta(x, y))$$

We choose α, β so that both are smaller than $\delta \|x, y\|$, with such a choice depending on an appropriate neighborhood. A point on notation, $\|x, y\|$ means $\|(x, y)\|$.

Lemma. If $\delta < \frac{\mu-\lambda}{2+\gamma+\frac{1}{\gamma}}$, then

$$Df_p H_p^\gamma \subseteq \mathrm{Int}(H_{f(p)}^\gamma), \quad Df_p^{-1} V_{f(p)}^\gamma \subseteq \mathrm{Int}(V_p^\gamma)$$

Proof of Lemma. Suppose that $(u, v) \in H_p^\gamma$, then we have $\|v\| \leqslant \gamma \|u\|$. Looking at the image of this vector, we have

$$(u', v') := Df_p(u, v) = (Au + D\alpha_p(u, v), Bv + D\beta_p(u, v))$$

We may estimate

$$\|v'\| \leqslant \|Bv\| + \|D\beta_p(u, v)\| < \lambda \|v\| + \delta \|u, v\|$$
$$\|u'\| \geqslant \|Au\| - \|D\alpha_p(u, v)\| > \mu \|u\| - \delta \|u, v\|$$

We want to verify $\|v'\| \leqslant \gamma \|u'\|$. This is just an arithmetic verification. For instance, we may use the Pythagorean theorem and say

$$\|u, v\| < (1 + \gamma) \|u\|$$

Now, we have

$$\lambda\gamma\|u\| + \delta(1+\gamma)\|u\| \leqslant \gamma(\mu\|u\| - \delta(1+\gamma)\|u\|)$$

Double check what this inequality should say - handwriting unclear. This concludes the lemma, and we see that for sufficiently small δ, we send our cone to another cone. □

Our next step is to show that vectors in horizontal cones expand and vectors in vertical cones contract.

Lemma. If $(u,v) \in H_p^\gamma$, then

$$\|Df(u,v)\| > \left(\frac{\mu}{1+\gamma} - \delta\right)\|u,v\|$$

If $(u,v) \in V_p^\gamma$, then

$$\|Df(u,v)\| < (1+\gamma)(\lambda+\delta)\|u,v\|$$

Proof of Lemma. We have

$$\begin{aligned}\|u',v'\| \geqslant \|u'\| \geqslant \|Au\| - \|D\alpha(u,v)\| &> \mu\|v\| - \delta(1+\gamma)\|u\| \\ &> \frac{\mu - \delta(1+\gamma)}{1+\gamma} \cdot \|u,v\|\end{aligned}$$

□

We now define our distributions at each point p. As a reminder, E^+ will be k-dimensional and E^- will be $n-k$ dimensional.

$$(E^+)_p = \bigcap_{j=0}^{\infty} Df_{f^{-1}(p)} \ldots Df_{f^{-j}(p)} H^\gamma_{f^{-j}(p)}$$

$$(E^-)_p = \bigcap_{j=0}^{\infty} Df_p^{-1} \ldots Df^{-1}_{f^{j-1}(p)} V^\gamma_{f^j(p)}$$

These will be our invariant distributions. Define a space

$$S_j = Df_{f^{-1}(p)} \ldots Df_{f^{-j}(p)}(\mathbb{R}^n \times \{O\})$$

Here is the image of the axis of some cone. Notice that $S_j \in T_p(\mathbb{R}^n)$. If $\dim(S_j)$ for all j, then there is a limit $S_j \xrightarrow[j\to\infty]{} S$. We claim that $S = E_p^+$. Call $\mathscr{P} := Df_{f^{-1}(p)} \ldots Df_{f^{-i}(p)}$. It follows that

$$\mathscr{P}(U_i, v_i) = \mathscr{P}(u_i, 0) + \mathscr{P}(0, v_i)$$

We also know that $\mathscr{P}(u_i, 0) \in S_i$ and $\mathscr{P}(0, v_i) \in V$ and by Lemma 2, it is small. For all $(u,v) \in E_p^+$ and there exists an N such that for any $i > N$, there exists (u_i, v_i) such that

$$(u,v) = \mathscr{P}_i(u_i, v_i) \in U(S_i, \varepsilon)$$

where U indicates the ε-neighborhood of S_i. This means that any point of E_p^+ is close to a point of S_i. The converse direction also holds. From this and by sending $\varepsilon \to 0$, we see that $E_p^+ = S$.

We claim that E^+ and E^- depend continuously on a point p. Check 1:12:00 in the Lecture for more details, which are given geometrically. The key conclusion is that any two linear subspaces into these cones are ε close, and this is why these distributions are continuous.

As previously mentioned, all S_j tend to the same limit space. It turns out that these distributions are the only invariant distributions.

Lemma. E^+ and E^- are the only invariant distributions in H^γ, V^γ.

We continue to Step 2, where our goal is to obtain invariant graphs. Our setting is the space of Lipschitz functions and we define

$$C_\gamma^0 := \{\varphi^+ : \mathbb{R}^k \to \mathbb{R}^{n-k} \mid \text{Lipschitz}, \text{Lipschitz constant } \gamma, \varphi^+(0) = 0\}$$

where $+$ corresponds to the direction of time. We also have the function $f : \mathbb{R}^n \to \mathbb{R}^n$ which will act on the graphs of the function in the set above. We would like to find functions satisfying $f(\text{graph}(\varphi^+)) = \text{graph}(\varphi^+)$.

Lemma. Let $\psi \in C_\gamma^0$. We have $f(\text{graph}(\varphi)) = \text{graph}(\psi)$

Proof of Lemma. Consider a map $G_\varphi : \mathbb{R}^k \to \mathbb{R}^k$ defined by

$$G_\varphi(x) = Ax + \alpha(x, \varphi(x))$$

We need to prove that this map results in a graph. In fact we would like to show that $G_\varphi(x)$ is a bijection. In other words, for every x_0 in the codomain, there exists a unique x such that $G_\varphi(x) = x_0$. Resume at 1:25:41.