Differential Geometry Exercises

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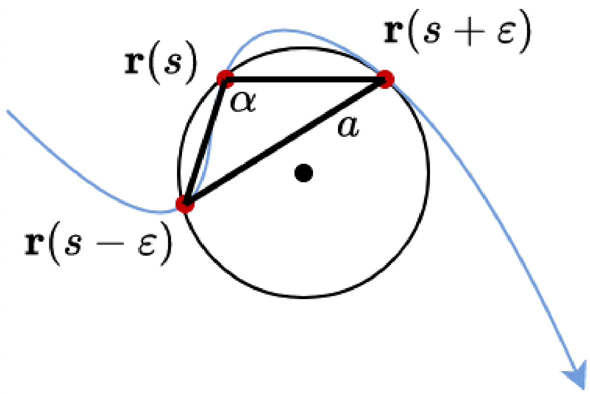
Curvature

Class 1

Problem 1. Let $\mathbf{r}(s)$ be a planar curve parameterized by a natural parameter s. Consider the circle passing through points $\mathbf{r}(s-\varepsilon), \mathbf{r}(s), \mathbf{r}(s+\varepsilon)$ and denote its radius by $R(s,\varepsilon)$. Find the limit

$$\lim_{\varepsilon\to 0} R(s,\varepsilon)$$

Observe that this result provides an alternative approach for defining the curvature of a planar curve.
We provide a drawing of the situation described above, where α an angle at the point $\mathbf{r}(s)$ and a is the length of the opposite side.



By the Law of Sines, we have the relation

$$R = \frac{a}{2\sin\alpha}$$

Since we want to determine a limit expression for R, it is useful to think in terms of the Taylor series approximation. Recall the Taylor series approximation for a function f around the point x_0:

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2 + \ldots = \sum_{n=0}^{\infty} \frac{f^n(x_0)(x-x_0)^n}{n!}$$

Making the substitution $x - x_0 = \varepsilon$, we have

$$f(x_0+\varepsilon) = f(x_0) + f'(x_0)\varepsilon + \frac{1}{2}f''(x_0)\varepsilon^2 + O(\varepsilon^3)$$

This is a useful form when we consider taking expression in the limit as $\varepsilon \to 0$. Thus, we see that as $\varepsilon \to 0$

$$r(s+\varepsilon) = r(s) + \varepsilon\dot{r}(s) + \frac{\varepsilon^2}{2}\ddot{r}(s) + O(\varepsilon^3)$$
$$r(s-\varepsilon) = r(s) - \varepsilon\dot{r}(s) + \frac{\varepsilon^2}{2}\ddot{r}(s) + O(\varepsilon^3)$$

Observe that

$$(r(s+\varepsilon)-r(s))\times(r(s-\varepsilon)-r(s)) = |r(s+\varepsilon)-r(s)||r(s-\varepsilon)-r(s)|\sin\alpha$$

Making substitutions using the Taylor approximation, we have

$$\left[\varepsilon\dot{r}(s)+\frac{\varepsilon^2}{2}\ddot{r}(s)+O(\varepsilon^3)\right]\times\left[-\varepsilon\dot{r}(s)+\frac{\varepsilon^2}{2}\ddot{r}(s)+O(\varepsilon^3)\right] = \varepsilon^3|\dot{r}(s)||\ddot{r}(s)|+O(\varepsilon^3)$$

$$\sin\alpha = \left[\varepsilon^3|\dot{r}(s)||\ddot{r}(s)|+O(\varepsilon^3)\right]\cdot\left[\left|\varepsilon\dot{r}(s)+\frac{\varepsilon^2}{2}\ddot{r}(s)+O(\varepsilon^3)\right|\left|-\varepsilon\dot{r}(s)+\frac{\varepsilon^2}{2}\ddot{r}(s)+O(\varepsilon^3)\right|\right]^{-1}$$

Keeping in mind that negatives cancel since the cross product is *not commutative*. Also

$$a = |r(s+\varepsilon)-r(s-\varepsilon)| = |2\varepsilon\dot{r}(s)+O(\varepsilon^3)|$$

Finally, by substitution

$$\begin{aligned} R(s,\varepsilon) &= \frac{a}{2\sin\alpha} \\ &= |2\varepsilon\dot{r}(s)|\cdot\left[\left|\varepsilon\dot{r}(s)+\frac{\varepsilon^2}{2}\ddot{r}(s)+O(\varepsilon^3)\right|\left|-\varepsilon\dot{r}(s)+\frac{\varepsilon^2}{2}\ddot{r}(s)+O(\varepsilon^3)\right|\right]\cdot\left[2\varepsilon^3|\dot{r}(s)||\ddot{r}(s)|+O(\varepsilon^3)\right]^{-1} \\ &= |\dot{r}(s)|\left|\dot{r}(s)+\frac{\varepsilon}{2}\ddot{r}(s)+O(\varepsilon^2)\right|\left|-\dot{r}(s)+\frac{\varepsilon}{2}\ddot{r}(s)+O(\varepsilon^2)\right|\cdot\left[|\dot{r}(s)||\ddot{r}(s)|+O(1)\right]^{-1} \end{aligned}$$

Taking the limit, we see that

$$\lim_{\varepsilon\to 0} R(s,\varepsilon) = \frac{|\dot{r}(s)|^2}{|\ddot{r}(s)|} = \frac{1}{|\ddot{r}(s)|}$$

This shows that the inverse of the curvature is the radius of the circle. So we may define curvature as the inverse of the radius to the osculating circle to the point on that curve. At the point of tangency, both the direction of the tangent vector and curvature of the circle match those of the curve.

Problem 2. Suppose we have a planar curve $r(s) = (x(s), y(s))$ with zero curvature and that s is a natural parameter. Since s is a natural parameter, we know that $\left|\frac{dr}{ds}\right| = 1$. Since $r(s)$ has zero curvature, we know that $\left|\frac{d^2r}{ds^2}\right| = 0$. In other words, we have

$$\begin{aligned} \dot{x}(s)^2+\dot{y}(s)^2 &= 1 \\ \ddot{x}(s)^2+\ddot{y}(s)^2 &= 0 \end{aligned}$$

A general solution that satisfies the first equation is $\dot{x}(s) = \cos(\theta(s))$ and $\dot{y}(s) = \sin(\theta(s))$, where $\theta(s)$ is some differentiable function of s. Taking the derivative, we see that

$$\ddot{x}(s) = -\sin(\theta(s))\cdot\dot{\theta}(s) = 0 \text{ and } \ddot{y}(s) = \cos(\theta(s))\cdot\dot{\theta}(s) = 0$$

Since $\dot{\theta}(s) = 0$, this implies that $\theta(s)$ is a constant which we may represent simply as θ. Then we integrate $\dot{x}(s)$ and $\dot{y}(s)$ to find

$$r(s) = (\cos(\theta)s + C_1, \sin(\theta)s + C_2)$$

which is precisely a straight line in the direction of $\cos(\theta), \sin(\theta)$. $\square$

Problem 3. Suppose we have a planar curve with constant curvature $k \neq 0$. Since s is a natural parameter, we must have the same general form as in Problem 2. Further, we know that

$$\dot{\theta}(s)^2 = \ddot{x}(s)^2 + \ddot{y}(s)^2 = k^2$$

As curvature is positive, we have $\dot{\theta}(s) = k$ and $\theta(s) = ks$. By substitution, we see that

$$\dot{x}(s) = \cos(ks) \text{ and } \dot{y}(s) = \sin(ks)$$

Integrating each component, we have

$$r(s) = \left(\frac{1}{k}\sin(ks), -\frac{1}{k}\cos(ks)\right)$$

This is the parametrization of a circle with radius $\frac{1}{k}$. $\square$

Problem 4. Let $\mathbf{r}(t) = (x(t), y(t))$ be a regular curve in $\mathbb{R}^2$, where t is an *arbitrary* parameter. Prove that one can find the curvature using the following formula

$$k(t) = \frac{|\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)|}{|\dot{\mathbf{r}}(t)|^3}$$

Remark that one can rewrite the formula above as

$$k(t) = \frac{|\ddot{x}(t)\dot{y}(t) - \ddot{y}(t)\dot{x}(t)|}{(\dot{x}(t)^2 + \dot{y}(t)^2)^{\frac{3}{2}}}$$

Solution. First, we prove this for the case where s is a natural parameter. If s is a natural parameter, we know that $(\dot{\mathbf{r}}, \ddot{\mathbf{r}}) = 0$. Thus, we see that

$$|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}| = |\dot{\mathbf{r}}||\ddot{\mathbf{r}}| \sin\left(\frac{\pi}{2}\right) = |\ddot{\mathbf{r}}| = k(s)$$

We conclude that

$$k(s) = \frac{|\dot{\mathbf{r}}(s) \times \ddot{\mathbf{r}}(s)|}{|\dot{\mathbf{r}}(\mathbf{s})|^3}$$

Now, suppose that t is an arbitrary parameter. Assume that we have s as a function of t, this is always possible by considering the arc length function. Thus, we have our reparametrization $\mathbf{r}(t) = \mathbf{r}(s(t))$ and by the chain rule

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} \text{ and } \frac{d^2\mathbf{r}}{dt^2} = \frac{d^2\mathbf{r}}{ds^2}\left(\frac{ds}{dt}\right)^2 + \frac{d\mathbf{r}}{ds}\frac{d^2s}{dt^2}$$

We consider the numerator first

$$|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}| = \left|\frac{d\mathbf{r}}{ds}\frac{ds}{dt} \times \left(\frac{d^2\mathbf{r}}{ds^2}\left(\frac{ds}{dt}\right)^2 + \frac{d\mathbf{r}}{ds}\frac{d^2s}{dt^2}\right)\right| = \left|\frac{ds}{dt}\right|^3 \left|\frac{d\mathbf{r}}{ds} \times \frac{d^2\mathbf{r}}{ds}\right|$$

where we recall that the vector product of vectors in the same direction is 0. The denominator is simply $|\dot{\mathbf{r}}(t)|^3 = \left|\frac{ds}{dt}\right|^3 \cdot \left|\frac{d\mathbf{r}}{ds}\right|^3$ Thus, we see that with an arbitrary parameter t, the expression simplifies to the case where we have a natural parameter s. $\square$

Note that the second formula is simply the first written in coordinates.

Problem 5. Remark that the Frenet frame is not defined at the points of a curve where its curvature is zero. But if we consider an oriented plane we can define the Frenet frame in the following way: define the oriented normal vector n_o as such a vector that v, n_o is a positively oriented orthonormal basis. Then we can define oriented curvature k_o as such a quantity that $\dot{v} = k_o n_o$. Provide an example where the oriented curvature k_0 is negative and $k_0 \neq k$. Find $\dot{n}_o$ and write oriented Frenet formulae.

Solution. Consider the curve $r(t) = (t, t^3)$. Taking derivatives, we have

$$\dot{r}(t) = (1, 3t^2)$$
$$\ddot{r}(t) = (0, 6t)$$

Based on our previous formula, we know that the curvature is given by

$$k(t) = \frac{|\ddot{x}(t)\dot{y}(t) - \ddot{y}(t)\dot{x}(t)|}{(\dot{x}(t)^2 + \dot{y}(t)^2)^{\frac{3}{2}}} = \frac{|0 - 6t|}{(1 + 9t)^{\frac{3}{2}}} = \frac{|6t|}{(1 + 9t)^{\frac{3}{2}}}$$

Notice that when $t > 0$, then $v, \ddot{r}$ is oriented in the same way as v, n_0 and in the opposite way when $t < 0$. We can understand orientation in the same direction based on if the angle between the two vectors belongs to the range $0 \leq \theta \leq \pi$ or not. Thus, we see that

$$n_0 = \operatorname{sgn}(t) \cdot n \text{ and } k_0 = \operatorname{sgn}(t) \cdot k$$

Class 2

Problem 1. Prove that a curve in $\mathbb{R}^3$ with zero curvature is a straight line. Prove that a curve in $\mathbb{R}^3$ with zero torsion is a planar curve, i.e. belongs to a plane in $\mathbb{R}^3$.

Solution. Suppose we have a natural parameter s and curvature $k = 0$. By definition, we have

$$\left|\frac{d^2 r}{ds^2}\right| = \sqrt{\ddot{x}(s)^2 + \ddot{y}(s)^2 + \ddot{z}(s)^2} = 0$$

Thus, we have the system of linear differential equations

$$\ddot{x}(s) \equiv 0 \quad \ddot{y}(s) \equiv 0 \quad \ddot{z}(t) \equiv 0$$

Solving the system, we have

$$r(s) = (a_1 s + b_1, a_2 s + b_2, a_3 s + b_3)$$

which is precisely the equation of a line. Suppose now that a curve r has zero torsion. In a geometric sense, the torsion measures how quickly the osculating plane changes. We define the function

$$f(s) := (r(s) - r(s_0), b(s))$$

note that the above means the scalar product of $r(s) - r(s_0)$ and b. Since torsion $\varkappa = 0$, from our Frenet equations, we know that $0 = \dot{b} = -\varkappa n$. This means that our binormal vector must be constant, so $b(s) \equiv b_0$. Further, we know that $f(s_0) = 0$ and taking the derivative, we have

$$f'(s) = (r'(s), b_0) + (r(s), 0) = (v(s), b_0) = 0$$

Now, let us denote

$$b_0 := (A, B, C) \quad r(s_0) = (x_0, y_0, z_0)$$

which means

$$\begin{aligned} 0 \equiv f(s) &= \langle (x(s) - x_0, y(s) - y_0, z(s) - z_0), (A, B, C) \rangle \\ &= Ax(s) + By(s) + Cz(s) + D \end{aligned}$$

where $D = -x_0 t - y_0 B - z_0 C$. Since our curve must satisfy the equation of this plane and since A, B, C, D are constants, we see that this is a planar curve.

□

Problem 2. Let $\mathbf{r}(t) = x(t)e_1 + y(t)e_2 + z(t)e_3 = (x(t), y(t), z(t))$ be a regular curve in $\mathbb{R}^3$, where t is an arbitrary parameter. Prove that one can find the curvature using the formula

$$k(t) = \frac{|\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)|}{|\dot{\mathbf{r}}(t)|^3},$$

and one can find the torsion using the formula

$$\varkappa = \frac{(\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}})}{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2},$$

where $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ denotes the oriented volume of the parallelepiped generated by the vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$.

Solution. We will prove the equation for a natural parameter, and then extend this to an arbitrary parameter. Let s be a natural parameter for r. We know that $\dot{r} = v$, $\ddot{r} = \dot{v} = kn$ and since s is natural, $|\dot{r}|^3 = 1$. We have proven the equation for curvature, in the past exercise session, as review:

$$\frac{|\dot{r} \times \ddot{r}|}{|\dot{r}|} = \frac{|v \times kn|}{1} = k|b| = k$$

Now, take an arbitrary parameter t. Suppose $r(s(t))$. This parametrization is always possible because arc length is a natural parameter. By the chain rule, we have

$$\frac{dr}{dt} = \frac{dr}{ds}\frac{ds}{dt}$$

After making this substitution, we will see that the formula for curvature reduces to that of the expression we obtained previously using the natural parametrization. The derivation does not change when r is a space curve rather than a planar curve, because the Frenet formula for velocity does not change. Note that

$$\begin{pmatrix} v \\ n \\ b \end{pmatrix}' = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \varkappa \\ 0 & -\varkappa & 0 \end{pmatrix} \begin{pmatrix} v \\ n \\ b \end{pmatrix}$$

We apply a similar approach to the formula for torsion. First, we prove that the formula is true for a natural parameter, so

$$\dot{r} = v, \quad \ddot{r} = kn, \quad \dddot{r} = \ddot{v} = \dot{k}n + k\dot{n}$$

To further simplify $\ddot{v}$, we must use our second Frenet equation

$$\begin{aligned} \ddot{v} &= \dot{k}n + k\dot{n} \\ &= \dot{k}n - k^2 v + k\varkappa b \end{aligned}$$

We recall some properties of the oriented volume.

1. $(a, b, c_1 + c_2) = (a, b, c_1) + (a, b, c_2)$
2. $(a, b, lc) = l(a, b, c)$
3. $(a, b, a) = 0$

These are just the usual properties of determinants. Now, we compute

$$\begin{aligned} (\dot{r}, \ddot{r}, \dddot{r}) &= (v, kn, \dot{k}n - k^2 v + k\varkappa b) \\ &= (v, kn, k\varkappa b) \\ &= k^2 \varkappa \end{aligned}$$

We have proven previously that $|\dot{r} \times \ddot{r}| = k$ when the parameter is natural. Thus

$$\frac{(\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}})}{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2} = \frac{k^2 \varkappa}{k^2} = \varkappa$$

as claimed.
Next, suppose that we have an arbitrary parameter now. By the chain rule, we have

$$\frac{dr}{dt} = \frac{dr}{ds}\frac{ds}{dt} \quad \frac{d^2r}{dt^2} = \frac{d^2r}{ds^2}\left(\frac{ds}{dt}\right)^2 + \frac{dr}{ds}\frac{d^2s}{dt^2}$$
$$\frac{d^3r}{dt^3} = \frac{d^3r}{ds^3}\left(\frac{ds}{dt}\right)^3 + c_1\frac{d^2r}{ds^2} + c_2\frac{dr}{ds}$$

Using the properties of oriented volume, we see that

$$\begin{aligned} \left(\frac{dr}{dt}, \frac{d^2r}{dt^2}, \frac{d^3r}{dt^3}\right) &= \left(\frac{dr}{ds}\frac{ds}{dt}, \frac{d^2r}{ds^2}\left(\frac{ds}{dt}\right)^2 + \frac{dr}{ds}\frac{d^2s}{dt^2}, \frac{d^3r}{ds^3}\left(\frac{ds}{dt}\right)^3 + c_1\frac{d^2r}{ds^2} + c_2\frac{dr}{ds}\right) \\ &= \left(\frac{dr}{ds}\frac{ds}{dt}, \frac{d^2r}{ds^2}\left(\frac{ds}{dt}\right)^2, \frac{d^3r}{ds^2}\left(\frac{ds}{dt}\right)^3\right) \\ &= \left(\frac{ds}{dt}\right)^6 \left(\frac{dr}{ds}, \frac{d^2r}{ds^2}, \frac{d^3r}{ds^3}\right) \end{aligned}$$

We also recall from a previous computation:

$$|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}| = \left| \frac{d\mathbf{r}}{ds}\frac{ds}{dt} \times \left(\frac{d^2\mathbf{r}}{ds^2}\left(\frac{ds}{dt}\right)^2 + \frac{d\mathbf{r}}{ds}\frac{d^2 s}{dt^2} \right) \right| = \left|\frac{ds}{dt}\right|^3 \left| \frac{d\mathbf{r}}{ds} \times \frac{d^2\mathbf{r}}{ds} \right|$$

Hence, we see that the case with an arbitrary parameter reduces to the case involving a natural parameter.

□

Problem 3. As you already know, a curve in $\mathbb{R}^3$ is defined by its positive curvature and its torsion up to an isometry of the ambient space. Given functions $f > 0$ and g, we can then write equations (called natural equations of a curve)

$$k(s) = f(s), \quad \varkappa(s) = g(s).$$

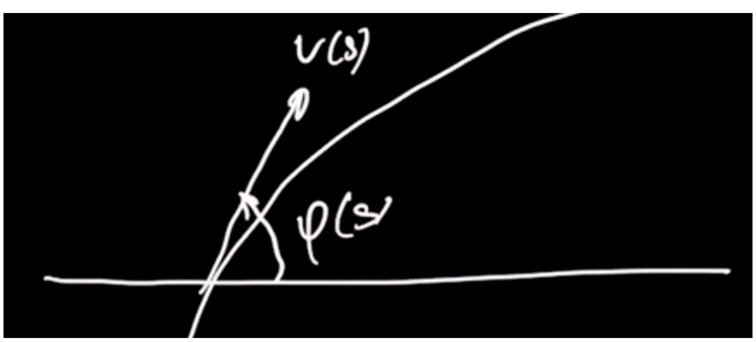
In the planar case we have just one equation

$$k(s) = f(s)$$

In the planar case solve a natural equation $k = k_0$, where $k_0 > 0$ is a constant.
Solution. We want to solve the natural equation $k(s) = k_0$. Solving the natural equation means we must find a curve $r(s)$ such that its curvature $k(s) = f(s)$. Again, since we are dealing with a planar curve, we only have one equation as our constraint.
From a previous problem, we know that a curve with positive constant curvature must be a circle with radius $\frac{1}{k_0}$. By the fundamental theorem of the local theory of curves, since a circle is a solution to this natural equation, then the uniqueness statement shows that all other solutions must only differ from the circle by an isometry.

□

Problem 4. Using integration, find a solution of a general natural equation $k(s) = f(s)$ for a planar curve.
Solution. Since s is a natural parameter, we know that $|v(s)| = 1$. Let $\varphi(s)$ be the angle that $v(s)$ makes with the x-axis at the point on the curve, $r(s)$.



Then, we may write $v(s) = (\cos(\varphi(s)), \sin(\varphi(s)))$. Recall that curvature is the angular velocity of the rotation of the tangent line, thus

$$\dot{\varphi}(s) = k(s)$$

But then, by integration we know that

$$\varphi(s) = \int k(s)\,ds + \varphi_0$$

By substitution, we see that

$$\dot{r}(s) = v(s) = \left(\cos\left(\int k(s)\,ds + \varphi_0\right), \sin\left(\int k(s)\,ds + \varphi_0\right)\right)$$
$$r(s) = \left(\int \cos\left(\int k(s)\,ds + \varphi_0\right) + x_0, \int \sin\left(\int k(s)\,ds + \varphi_0\right) + y_0\right)$$

Notice that we have three constants here φ_0, x_0, y_0. The constants x_0, y_0 relate to translations and φ_0 is the constant which determine the rotation of the curve (these are exactly the isometries). Reflection may be obtained by considering the parameter $-s$. This is another way to prove the fundamental theorem of the local theory of curve in the plane.
Note that analogous formulas do not exist for the 3-dimensional case.

□

Problem 5. Let a curve with $k \neq 0$ and $\varkappa \neq 0$ belong to a sphere of radius R. Prove that

$$R^2 = \frac{1}{k^2}\left(1 + \frac{(k')^2}{(\varkappa k)^2}\right)$$

where $'$ denotes the derivative with respect to a natural parameter.
Solution. Recall the Frenet basis, $v(s), n(s), b(s)$. Since it is a basis, we may write any vector in $\mathbb{R}^3$ as a linear combination of these vectors. In particular, for a curve r lying on a sphere of radius R, we may write

$$r(s) = \alpha(s)v(s) + \beta(s)n(s) + \gamma(s)b(s)$$

Because v, n, b is an orthonormal basis, we know that

$$R^2 = |r(s)|^2 = \alpha(s)^2 + \beta(s)^2 + \gamma(s)^2$$

The first equality is because the length of the position vector to any point on the sphere is equivalent to the radius. Observe that

$$(r, r) = R^2$$
$$2(r, \dot{r}) = 0$$

Thus, $(r, v) = 0$. Taking the scalar product on both sides by $v(s)$, we see

$$0 = (r, v) = \alpha(v, v) + \beta(n, v) + \gamma(b, v) = \alpha$$

Differentiating the scalar product once more, we have

$$0 \equiv (r, v)' = (\dot{r}, v) + (r, \dot{v}) = 1 + (r, kn) = 1 + k(r, n) = 1 + k\beta$$

So $\beta = -\frac{1}{k}$. Differentiating the scalar product a final time, we see

$$(1 + (r, kn))' = 0 + (\dot{r}, kn) + (r, \dot{k}n + k\dot{n})$$

From the Frenet equations, we know that $\dot{n} = -kv + \varkappa b$ and by substitution:

$$\begin{aligned}
(\dot{r}, kn) + (r, \dot{k}n + k\dot{n}) &= \underbrace{(\dot{r}, kn)}_{(v,kn)=0} + (r, \dot{k}n + k(-kv + \varkappa b)) \\
&= (r, \dot{k}n) - (r, k^2 v) + (r, k\varkappa b) \\
&= \dot{k}\beta - k^2\alpha + k\varkappa\gamma \equiv 0
\end{aligned}$$

After substituting for $\beta = \frac{1}{k}$, we see that

$$k\varkappa\gamma = \frac{\dot{k}}{k} \implies \gamma = \frac{\dot{k}}{k^2\kappa}$$

We conclude that

$$R^2 = \alpha^2 + \beta^2 + \gamma^2 = \frac{1}{k^2} + \left(\frac{\dot{k}}{k^2\varkappa}\right)^2 = \frac{1}{k^2}\left(1 + \frac{(k')^2}{(\varkappa k)^2}\right)$$

□

Surfaces

Class 3: First Fundamental Form

Recall that a surface is a map $r : U \to \mathbb{R}^3$ where $U \subseteq \mathbb{R}^2$ such that

1. $r \in C^\infty(\mathbb{R}^2, \mathbb{R}^3)$
2. r is regular, meaning the Jacobian matrix has full rank.

The second definition relates to the idea of natural parametrization. For surfaces, there is no direct analogue of the natural parametrization on the surface, leading to the concept of the **first fundamental form.**

$$\mathrm{I} = \begin{pmatrix} (r_u, r_u) & (r_u, r_v) \\ (r_u, r_v) & (r_v, r_v) \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

Consider the entries of the first fundamental form, which are scalar products indicated by

$$g_{ij} = \left(\frac{\partial r}{\partial x^i}, \frac{\partial r}{\partial x^j}\right)$$

In one dimension, there is only one entry in the matrix, g_{11}, corresponding to the length of the velocity vector. In a curve, we can get rid of it by choosing a natural parameter, but this is not possible for surfaces.

Problem 1. Find a parametrization $\mathbf{r}(\varphi, \psi)$ of a torus of revolution. Find basis vectors $\mathbf{r}_\varphi$ and $\mathbf{r}_\psi$ of the tangent plane at an arbitrary point of the torus. Find the first quadratic (fundamental) form for the torus.

Solution. If we consider the circular rotation of the torus in the xz-plane and xy-plane separately, we have the parametrization

$$r(\varphi, \psi) = ((r\cos\psi + R)\cos\varphi, (r\cos\psi + R)\sin\varphi, r\sin\psi)$$

where R specifies the radius of the hole in the torus. Finding r_φ and r_ψ along with the first fundamental form is just a computation

$$I = \begin{pmatrix} (r_\varphi, r_\varphi) & (r_\varphi, r_\psi) \\ (r_\varphi, r_\psi) & (r_\psi, r_\psi) \end{pmatrix} = \begin{pmatrix} (r\cos\psi + R)^2 & 0 \\ 0 & r^2 \end{pmatrix}$$

we will omit these details. Notice that I is independent of ψ.

□

Problem 2. Let

$$\vec{\mathbf{r}} = (u\cos v, u\sin v, av)$$

be a parametric surface. Find the angle of intersection of curves

$$u + v = 0, \quad \text{and} \quad u - v = 0.$$

Solve this problem twice, using the local coordinates u, v and using the global coordinates x, y, z.

Solution. This surface is actually a helicoid. We will parametrize two curves in local coordinates

$$\begin{cases} u_1(t_1) = t_1 \\ v_1(t_1) = -t_1 \end{cases} \qquad \begin{cases} u_2(t_2) = t_2 \\ v_2(t_2) = t_2 \end{cases}$$

The system of equations $u + v = 0, u - v = 0$ has $(0, 0)$ as its only solution, so this is the point of intersection. We begin computing entries in the first fundamental form

$$\begin{aligned}(r'(t_1), r'(t_2)) &= \left(\frac{\partial r}{\partial u}\dot{u}_1 + \frac{\partial r}{\partial v}\dot{v}_1, \frac{\partial r}{\partial u}\dot{u}_2 + \frac{dr}{dv}\dot{v}_2\right) \\ &= \dot{u}_1\dot{u}_2\left(\frac{\partial r}{\partial u}, \frac{\partial r}{\partial u}\right) + (\dot{u}_1\dot{v}_2 + \dot{u}_2\dot{u}_1)\left(\frac{\partial r}{\partial u}, \frac{\partial r}{\partial v}\right) + \dot{u}_2\dot{v}_2\left(\frac{\partial r}{\partial v}, \frac{\partial r}{\partial v}\right) \\ &= \begin{pmatrix} \dot{u}_1 & \dot{v}_1 \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \dot{u}_2 \\ \dot{v}_2 \end{pmatrix}\end{aligned}$$

Calculating the first fundamental form, we have

$$\mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & u^2 + a^2 \end{pmatrix}$$

Calculating the derivatives

$$\begin{cases} \dot{u}_1(t_1) = 1 \\ \dot{v}_1(t_1) = -1 \end{cases} \qquad \begin{cases} \dot{u}_2(t_2) = 1 \\ \dot{v}_2(t_2) = 1 \end{cases}$$

Working with curves in the uv-plane is very convenient. Now, we need to compute $(r'(t_1), r'(t_2))$ only at the point $(0,0)$. At this point, our first fundamental form is

$$\mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix}$$

Then our formula states

$$(r'(t_1), r'(t_2)) = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 - a^2$$

To compute the angle, we have

$$\cos\alpha = \frac{(r'(t_1), r'(t_2))}{|r'(t_1)||r'(t_2)|} = \frac{1-a^2}{1+a^2}$$

This highlights one of the uses of the first fundamental form. We can use it to understand how two vectors in the local space are transformed when they are mapped to the global space.

The full approach in global coordinates is omitted for now, but we note that

$$r(u_1(t_1), v_1(t_1)) = r(t_1, -t_1) = (t_1\cos(-t_1), t_1\sin(-t_1), -at_1)$$

The idea is just to represent the curves in global space and take the scalar product in that space.

□

Problem 3. Let

$$\frac{R^2}{(1+u^2+v^2)^2} \begin{pmatrix} 1+v^2 & -uv \\ -uv & 1+u^2 \end{pmatrix}$$

be a metric (first quadratic form) on a surface. Find the length of a curve $u = v$.

Solution. If we parameterize the curve using another parameter t, we have

$$\begin{cases} u(t) = t \\ v(t) = t \end{cases}$$

for $t \in \mathbb{R}$. Then, the length of the curve is

$$\ell(t) = \int_{-\infty}^{\infty} |r'(t)|\,dt$$

Even though we do not have an explicit formula of the surface, knowing the first fundamental form is enough to finish this computation. We compute

$$\begin{aligned}(r'(t), r'(t)) &= \begin{pmatrix} \dot{u} & \dot{v} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} (1+u^2+v^2)^2 \begin{pmatrix} 1+v^2 & -uv \\ -uv & 1+u^2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{R^2}{(1+u^2+v^2)^2}\end{aligned}$$

Notice that although the left-hand side depends on t, the right hand side seems to not depend on t. But we may simply substitute $u(t) = t$ and $v(t) = t$ from our previous parametrization to see where the dependence lies. Making this substitution, we have

$$
\begin{aligned}
(r'(t), r'(t)) &= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1+t^2 & -t^2 \\ -t^2 & 1+t^2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{R^2}{(1+2t^2)^2} \\
&= \frac{2R^2}{(1+2t^2)^2}
\end{aligned}
$$

Thus,

$$\ell(t) = \int_{-\infty}^{\infty} \sqrt{2} \cdot \frac{R}{1+2t^2} \, dt = \pi R$$

Problem 4. Find the first quadratic form for a surface of revolution

$$\mathbf{r}(u, \varphi) = (x(u), \rho(u) \cos \varphi, \rho(u) \sin \varphi).$$

Solution. This is just a direct computation of I using the partial derivatives r_u, r_φ. We have

$$\mathrm{I} = \begin{pmatrix} x'(u)^2 + \rho'(u)^2 & 0 \\ 0 & \rho^2(u) \end{pmatrix}$$

□

Problem 5. Consider a surface defined by an implicit smooth equation

$$F(x, y, z) = 0$$

such that at any point of this surface grad $F \neq 0$. Prove that each point of the surface has a neighborhood such that there exists a regular smooth parametrization of this surface in this neighborhood.

Solution. Recall that the gradient is just the derivative map of a scalar-valued function (which may be in multiple variables).

$$\mathrm{grad} F(x_1, \dots, x_n) = \left(\frac{\partial F}{\partial x^1}, \dots, \frac{\partial F}{\partial x^n} \right)$$

Consider $F(x, y)$ and a point (x_0, y_0) such that $F(x_0, y_0) = 0$. By the premise, we know that $\mathrm{grad}(F(x_0, y_0)) \neq 0$. Then, by the implicit function theorem, there exists a neighborhood U such that $x_0 \in U$ and a smooth function $y(x)$ such that $F(x, y(x)) \equiv 0$ for all $x \in U$. Notice that for $r(x) = (x, y(x))$, we have $r'(x) = (1, y') \neq 0$.

Likewise, in the three dimensional case, consider some point (x_0, y_0, z_0) for which $F(x_0, y_0, z_0)$ and suppose without loss of generality that $\frac{\partial F}{\partial z}(x_0, y_0, z_0) \neq 0$. By the implicit function theorem, we may parameterize $z(x, y)$ and write

$$r(x, y) = (x, y, z(x, y))$$

Further, since

$$r_x = (1, 0, z_x) \text{ and } r_y = (0, 1, z_y)$$

we see that this is a locally regular and smooth surface in the neighborhood U of this point.

□

If we do not have the restriction $\mathrm{grad}(F) = 0$, we cannot say anything about the curve.

Class 4: Second Fundamental Form

Consider a surface $r(u, v)$. We know that r_u, r_v, m is a basis of $\mathbb{R}^3$. Because it is a basis, we may write

$$r_{uu} = \Gamma_{11}^1 r_u + \Gamma_{11}^2 r_v + Lm$$

where Γ_{ij}^p are the Christoffel symbols. How do we determine the coefficient in front of m? We consider projecting r_{uu} onto m

$$|\text{pr}_m r_{uu}| = \frac{(r_{uu}, m)}{|m|} = (r_{uu}, m) = L$$

Recall that for curves, we had $r_{ss} = 0 \cdot v + kn$. As an analogy, the second quadratic form is similar to "curvature" for the surfaces.

Problem 1. Find the second quadratic (fundamental) form, the Gaussian and the mean curvatures for a torus of revolution.

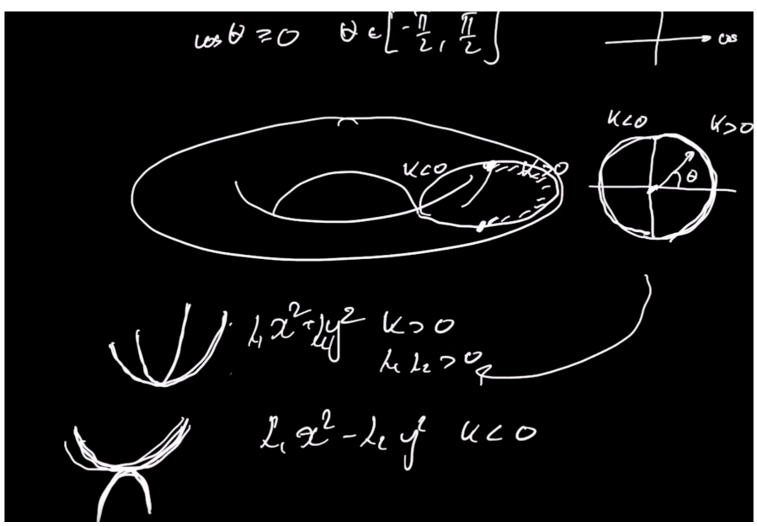
Solution. This is just a painful calculation. The first step is to find the unit normal vector to the tangent plane. Afterwards, we find the entries L, M, N of the second fundamental form. We are using the parametrization from the previous class. Skipping all the calculations, we find that

$$\text{II} = \begin{pmatrix} -(R + r\cos\theta)\cos\theta & 0 \\ 0 & -r \end{pmatrix}$$

In the basis of principal directions, the first fundamental form is the identity matrix and $\text{II} = \left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)$, where the λ_1 and λ_2 are the principal curvatures. Recall that $H = \lambda_1 + \lambda_2$ and $K = \lambda_1\lambda_2$. Further $K = \det(\text{I}^{-1}\text{II})$ and $H = \text{tr}(\text{I}^{-1}\text{II})$. Skipping computation again, we note

$$\text{I}^{-1}\text{II} = \begin{pmatrix} -\frac{\cos\theta}{R + r\cos\theta} & 0 \\ 0 & -\frac{1}{r} \end{pmatrix}$$

The Gaussian curvature is the product of the diagonal entries and the mean curvature is the sum. There is a geometrical meaning to K, depending on $K > 0$ or $K < 0$ we can have an elliptic or hyperbolic point, respectively.



□

Problem 2. Prove that a surface of zero Gaussian and mean curvature is a piece of plane.
Solution. Suppose $K \equiv 0$ and $H \equiv 0$. We remark that the second quadratic form is similar to the curvature of a curve. We proved previously that curves with curvature $k = 0$ are lines. Analogously, we are showing here that surfaces with $\mathrm{II} \equiv 0$ are planes. In the basis of principal directions w_1, w_2, we have $\mathrm{I} = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ and $\mathrm{II} = \left(\begin{smallmatrix}\lambda_1&0\\0&\lambda_2\end{smallmatrix}\right)$. However, we are given the system of equations that $\lambda_1 + \lambda_2 = 0$ and $\lambda_1\lambda_2 = 0$, so $\lambda_1 = 0$ and $\lambda_2 = 0$. thus, we see that II is equal to the zero matrix. Further any change of basis will result in the zero matrix as well. This means that in our basis r_u, r_v, the coefficients L, M, N are identically 0.
Now, if $r(u, v)$ is a plane, this means that the unit normal vector is constant, and so $m_u = 0, m_v = 0$. At each point, we have a basis r_u, r_v, m. Since $m_u = 0$, it is easy to see that it is orthogonal to r_u, r_v, m. However, is the converse true? Yes. Since r_u, r_v, m is a basis and m_u is orthogonal to each basis vector, then it must be orthogonal to every vector. So $m_u = 0$. The same applies to m_v.
So, we want to show $m_u = 0$. First, since $(m, m) = 1$, it follows that $(m_u, m) = 0$. Next, since $m = r_u \times r_v$, we see that $(r_u, m) = 0$. Differentiating, we have

$$\underbrace{(r_{uu}, m)}_{L=0} + (r_u, m_u) = 0$$

Similarly,

$$(r_v, m)_u = \underbrace{(r_{vu}, m)}_{M=0} + (r_v, m_u) = 0$$

Thus, since m_u is orthogonal to r_v, r_u, and m, it follows that $m_u = 0$. An analogous argument shows $m_v = 0$ and so we see that m is constant.
Now, consider a function $f(u, v) = (r(u, v) - r(u_0-, v_0), m)$. First, notice that $f(u_0, v_0) = 0$ because the first argument in the scalar product becomes 0. Next, we take the derivatives

$$\begin{aligned} f_u &= (r_u(u, v), m) + (r(u, v) - r(u_0, v_0), m_u) = 0 \\ f_v &= (r_v(u, v), m) + 0 = 0 \end{aligned}$$

Thus, $f \equiv 0$. Given some initial point $r_0 = r(u_0, v_0) = (x_0, y_0, z_0)$ and $m = (A, B, C)$, we see that

$$0 \equiv f(u, v) = (r - r_0, m) = A(x - x_0) + B(y - y_0) + C(z - z_0)$$

Therefore, if $\mathrm{II} = 0$, then we see that r is part of the plane (it may be some region in the plane).

□

Problem 3. Find the second quadratic form, the Gaussian and the mean curvatures for a surface of revolution

$$\mathbf{r}(u, \varphi) = (x(u), \rho(u)\cos\varphi, \rho(u)\sin\varphi).$$

Solution. Taking the derivatives

$$r_u = (x', \rho'\cos\varphi, \rho'\sin\varphi) \quad r_\varphi = (0, -\rho\sin\varphi, \rho\cos\varphi)$$

Thus, our first fundamental form (read about **coordinate lines** to understand more about meaning of when form is diagonal) is

$$\mathrm{I} = \begin{pmatrix} (x')^2 + (\rho')^2 & 0 \\ 0 & \rho^2 \end{pmatrix}$$

Skipping the calculation of the cross product

$$m = \frac{r_u \times r_\varphi}{|r_u \times r_\varphi|} = \frac{(\rho', x' \cos\varphi, -x' \sin\varphi)}{\sqrt{(x')^2 + (\rho')}^2}$$

Taking the second derivatives, we have

$$r_{\varphi\varphi} = (0, -\rho\cos\varphi, -\rho\sin\varphi) \quad r_{u\varphi} = (0, -\rho'\sin\varphi, \rho'\cos\varphi)$$

Thus

$$L = \frac{\rho' x'' - x'\rho''}{\sqrt{(x')^2 + (\rho')^2}}$$
$$M = 0$$
$$N = \frac{\rho x'}{\sqrt{(x')^2 + (\rho')^2}}$$

Now we compute $\mathrm{I}^{-1}\mathrm{II}$

$$\mathrm{I}^{-1}\mathrm{II} = \begin{pmatrix} \frac{\rho' x'' - x'\rho''}{((x')^2+(\rho')^2)^{\frac{3}{2}}} & 0 \\ 0 & \frac{x'}{\rho\sqrt{(x')^2+(\rho')^2}} \end{pmatrix}$$

So we just need the determinant and trace:

$$K = \frac{x'(\rho' x'' - x'\rho'')}{\rho((x')^2 + (\rho')^2)^2}$$
$$H = \frac{1}{((x')^2 + (\rho')^2)^{\frac{3}{2}}}(\rho\rho' x'' - \rho x'\rho'' + (x')^3 + x'(\rho')^2)$$

□

Problem 4. Prove that the only surfaces of revolution of zero mean curvature (such surfaces are called minimal) are a plane and a cathenoid, i. e. a surface obtained by rotation of a curve

$$\left(\frac{\cosh(at+b)}{a}, t\right)$$

Solution. Since we are considering a minimal surface, we know $H \equiv 0$ and from the equation in the previous problem

$$\rho\rho' x'' - \rho x'\rho'' + (x')^3 + x'(\rho')^2 = 0$$

If we consider the curve associated with slicing the surface through the xz-plane, then we have $(x(u), 0, p(u))$. We are considering this since we want to determine the shape of the curve. We may **locally** parameterize this as $(u, 0, p(u))$. Note that we may only make this change in parametrization when $x'(u) \neq 0$ in some interval. Then, we get

$$0 - \rho\rho'' + 1 + (\rho')^2 = 0$$

In the case where $x' \equiv 0$, then x is a constant, which means that our curve is a line and by rotating that, we get a part of a plane.
Now, returning to the previous case, we need to solve

$$-\rho\rho'' + 1 + (\rho')^2 = 0$$

Consider $\rho' = f(p)$, now

$$\rho'' = \frac{d}{du}\rho' = \frac{d}{du}f(\rho(u)) = \frac{df}{d\rho} \cdot \frac{d\rho}{du} = f \cdot \frac{df}{d\rho}$$

Plugging into our original equation, we have $-\rho f \frac{df}{d\rho} + 1 + f^2 = 0$. This is better because it is a first-order ODE where we can use separation of variables.

$$\frac{df}{d\rho} = \frac{1+f^2}{f\rho} \implies \frac{f}{1+f^2}df = \frac{d\rho}{\rho}$$

So, $f = \sqrt{c\rho^2 - 1}$. Since $\rho'(u) = f(p)$, we see that

$$\rho(u) = \frac{\cosh(\sqrt{c}u + b)}{\sqrt{c}}$$

□

Class 5: k-dimensional surfaces in $\mathbb{R}^n$

Recall the definition of the directional derivative

$$\partial_V f = \lim_{\varepsilon \to 0} \frac{f(A + \varepsilon V) - f(A)}{\varepsilon}$$

This is the derivative at A along the vector V for some $f : \mathbb{R}^n \to \mathbb{R}$. We know that

1. For $V = (V^1, \ldots, V^n)$

   $$\partial_V f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(A)V^i = \frac{\partial f}{\partial x^i}V^i$$

   where the last equality follows by Einstein summation convention using lower and upper indices.

2. Let $V \in T_A\Sigma$ and $\dim \Sigma = k$, and let $u^1, \ldots, u^k$ be local coordinates. Then we may write $V = (V^1, \ldots, V^k)$ in the basis $r_{u^1}, \ldots r_{u^k}$ of the tangent space. In other words, we may write every tangent vector using this basis. Letting $f : \Sigma \to \mathbb{R}$, $f(u^1, \ldots, u^k)$, we have the formula

   $$\partial_V f = \sum_{i=1}^{k} \frac{\partial f}{\partial u^i}(A)V^i = \frac{\partial f}{\partial u^i}V^i$$

   Even though $\Sigma \subseteq \mathbb{R}^n$, we recall that we may write $x^i(u^1, \ldots, u^k)$ for all $i \in \{1, \ldots, n\}$, so we may write $f\big|_\Sigma$ in the local coordinates $u^1, \ldots, u^k$.

Problem 1. Consider a circle $\mathbb{S}^1$ in the euclidean plane parametrized as

$$\mathbf{r}(\varphi) = (\cos\varphi, \sin\varphi).$$

Remark that at each point the corresponding tangent space is just a tangent line spanned by the vector $\mathbf{r}_\varphi$. Consider a function $f : \mathbb{S}^1 \longrightarrow \mathbb{R}$ defined in local coordinate φ as $f(\varphi) = \cos^2\varphi$, a point A with local coordinate $\varphi = \frac{\pi}{4}$ and a vector $V = 3\mathbf{r}_\varphi(A) \in T_A\mathbb{S}^1$.
a) Find $\partial_V f(A)$ using the local coordinate φ.
Since $T_A\mathbb{S}^1 = \operatorname{span}(r_\varphi)$, we see that $V = (V^1) = (3)$. We compute

$$\begin{aligned}\partial_V f &= \sum_{i=1}^{k} \frac{\partial f}{\partial u^i} \cdot V^i = \frac{df}{d\varphi} V^1 = 3\frac{df}{d\varphi}\bigg|_A = -2\cos\varphi\sin\varphi \cdot 3\bigg|_A \\ &= -3\sin 2\varphi\bigg|_A = -3\sin\frac{\pi}{2} = -3\end{aligned}$$

b) Now find $\partial_V f(A)$ using the global coordinates x, y. First, find global coordinates of the point A and the vector V. Next, extend f in a neighbourhood of a point A in $\mathbb{R}^2$ and write this extension in the global coordinates x, y. Finally, find $\partial_V f(A)$ and compare the result with the result obtained in a).
Observe that

$$r_\varphi = \frac{dr}{d\varphi} = (-\sin\varphi, \cos\varphi)$$

The the global coordinates of the vector are

$$3r_\varphi(A) = \left(\frac{-3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$$

How can we extend the function $f(\varphi)$ into the neighborhood of our manifold $r(\varphi) = (\cos\varphi, \sin\varphi)$? Take the extension $f(x, y) = x^2$ and notice that this works since $x^2\big|_{\mathbb{S}^1} = \cos^2\varphi$. Then, we compute

$$\partial_V f = \frac{\partial f}{\partial x} V^1 + \frac{\partial f}{\partial y} V^2\bigg|_A = -2x\frac{3}{\sqrt{2}}\bigg|_A = -2\frac{1}{\sqrt{2}}\frac{3}{\sqrt{2}} = -3$$

which matches our result in part (a).

□

Problem 2. Consider a unit sphere

$$\mathbf{r}(\varphi, \psi) = (\cos\varphi\sin\psi, \sin\varphi\sin\psi, \cos\psi).$$

Choose as the normal unit vector field the outward normal

$$\mathbf{m} = (\cos\varphi\sin\psi, \sin\varphi\sin\psi, \cos\psi).$$

Let $X = \mathbf{r}_\varphi, Y = \mathbf{r}_\psi$. Find the derivatives $\partial_X X, \partial_X Y, \partial_Y X$ and $\partial_X \mathbf{m}$. Find $B(X, X)$ and $B(X, Y)$.
Solution. Recall that for $\partial_X Y$, if $Y = (Y^1, Y^2, Y^3)$, then

$$\partial_X Y = (\partial_X Y^1, \partial_X Y^2, \partial_X Y^3)$$

It is important to notice that

$$\partial_{r_{u^1}} f = \frac{\partial f}{\partial u^1}$$

this is related to the proposition (on page 45 of the notes) that we discussed in class about taking the directional derivative in the direction of a tangent vector of a curve. Given r_{u^1}, its coordinates in the basis $r_{u^1}, \ldots, r_{u^k}$ are $V = (1, 0, \ldots, 0)$. This means that

$$\partial_V f = \sum_{i=1}^{k} \frac{\partial f}{\partial u^i} V^i = \frac{\partial f}{\partial u^1}$$

It is a special case of the second formula we discussed at the start of the class. This formula is useful as we may apply it directly

$$\partial_X X = \partial_{r_\varphi} X = \begin{pmatrix} \frac{\partial X^1}{\partial \varphi} \\ \frac{\partial X^2}{\partial \varphi} \\ \frac{\partial X^3}{\partial \varphi} \end{pmatrix}$$

Notice that

$$\begin{aligned} X = r_\varphi &= \frac{\partial}{\partial \varphi}(\cos\varphi \sin\psi, \sin\varphi \sin\psi, \cos\psi) \\ &= (-\sin\varphi \sin\psi, \cos\varphi \sin\psi, 0) \end{aligned}$$

Thus, we compute

$$\partial_X X = (-\cos\varphi \sin\psi, -\sin\varphi \sin\psi, 0)$$

Likewise

$$\begin{aligned} Y = r_\psi &= \frac{\partial}{\partial \psi}(\cos\varphi \sin\psi, \sin\varphi \sin\psi, \cos\psi) \\ &= (\cos\varphi \cos\psi, \sin\varphi \cos\psi, -\sin\psi) \end{aligned}$$

and

$$\begin{aligned} \partial_X Y &= \frac{\partial Y}{\partial \varphi} = (-\sin\varphi \cos\psi, \cos\varphi \cos\psi, 0) \\ \partial_Y X &= \partial_X Y = (-\sin\varphi \cos\psi, \cos\varphi \cos\psi, 0) \\ \partial_X m &= \frac{\partial m}{\partial \varphi} = (-\sin\varphi \sin\psi, \cos\varphi \sin\psi, 0) \end{aligned}$$

Now we need to find the second fundamental form, which we may compute by definition

$$\begin{aligned} B(X, Y) &= (\mathrm{Id} - P_A)(\partial_X Y) = (m, \partial_X Y)\, m \\ &= (\cos\varphi \sin\psi, \sin\varphi \sin\psi, \cos\psi) \cdot (-\sin\varphi \sin\psi, \cos\varphi \sin\psi, 0)\, m = \vec{0} \end{aligned}$$

We have simplified $(\mathrm{Id} - P_A)$ by recognizing that we are in $\mathbb{R}^3$ so our projection is onto a vector. Finally, we compute

$$B(X, X) = (\mathrm{Id} - P_A)(\partial_X X) = (m, \partial_X X) m = -\sin^2\psi \cdot \vec{m}$$

□

Note that II is the second fundamental form of a 2-dimensional surface in 3-dimensional space. On the other hand $B(X,Y) : T_A\Sigma \times T_A\Sigma \to N_A\Sigma$, which is also called the second fundamental form, is a vector. It turns out that $B(X,Y)$ is a generalization of II. We only need a scalar value in the case of 2-dimensional surfaces in $\mathbb{R}^3$ because the normal space is one-dimensional.

Problem 3. Let X and Y be two vector fields in $\mathbb{R}^n$.

a) Prove that there exists a vector field Z such that for any function φ one has

$$\partial_X \left(\partial_Y\right) \varphi - \partial_Y \left(\partial_X\right) \varphi = \partial_Z \varphi.$$

This vector field Z is called a commutator of vector fields X and Y and is denoted $[X,Y]$

Solution. Suppose we have the function $\varphi(x_1,\dots,x_n)$. By the chain rule, we have

$$\partial_X\varphi = \frac{\partial\varphi}{\partial x^i}X^i \qquad \partial_Y\varphi = \frac{\partial\varphi}{\partial x^j}Y^j$$

By substitution, we see that

$$\begin{aligned}\partial_X(\partial_Y\varphi) &= \partial_X\left(\frac{\partial\varphi}{\partial x^j}Y^j\right) = X^i\frac{\partial}{\partial x_i}\left(\frac{\partial\varphi}{\partial x^j}Y^j\right)\\ &= X^iY^j\frac{\partial^2\varphi}{\partial x^j x^i} + X^i\frac{\partial Y^j}{\partial x^i}\frac{\partial\varphi}{\partial x^j}\end{aligned}$$

Similarly, for the other order, we have

$$\partial_Y(\partial_X\varphi) = X^iY^j\frac{\partial^2\varphi}{\partial x^i x^j} + Y^j\frac{\partial X^i}{\partial x^j}\frac{\partial\varphi}{\partial x^i}$$

As a result, we see that

$$\partial_X(\partial_Y\varphi) - \partial_Y(\partial_X\varphi) = X^i\frac{\partial Y^j}{\partial x^i}\frac{\partial\varphi}{\partial x^j} - Y^j\frac{\partial X^i}{\partial x^j}\frac{\partial\varphi}{\partial x^i}$$

We may rename the indices on the second sum, replacing i with j, as it is a finite sum and the indices are both from $1,\dots,n$. So, we get

$$\begin{aligned}\partial_X(\partial_Y\varphi) - \partial_Y(\partial_X\varphi) &= X^i\frac{\partial Y^j}{\partial x^i}\frac{\partial\varphi}{\partial x^j} - Y^i\frac{\partial X^j}{\partial x^i}\frac{\partial\varphi}{\partial x^j}\\ &= \left(X^i\frac{\partial Y^j}{\partial x^i} - Y^i\frac{\partial X^j}{\partial x^i}\right)\frac{\partial\varphi}{\partial x^j}\\ &= \sum_{j=1}^n\left(\sum_{i=1}^n X^i\frac{\partial Y^j}{\partial x^i} - Y^i\frac{\partial X^j}{\partial x^i}\right)\frac{\partial\varphi}{\partial x^j}\\ &= \sum_{j=1}^n Z^j(x_1,\dots,x_n)\,\frac{\partial\varphi}{\partial x^j} = \partial_Z\varphi\end{aligned}$$

So, we have shown the existence directly.

□

b) Prove that

$$\partial_X Y - \partial_Y X = [X,Y],$$

i.e. prove that for an arbitrary smooth function φ one has

$$\partial_{\partial_X Y - \partial_Y X}\varphi = \partial_X(\partial_Y)\varphi - \partial_Y(\partial_X)\varphi.$$

Solution. From the previous problem, we saw that

$$Z^j(x_1, \ldots, x_n) = \left(\sum_{i=1}^{n} X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}\right)$$

Observe that

$$\partial_X Y - \partial_Y X = \frac{\partial Y}{\partial x_i} X^i - \frac{\partial X}{\partial x_i} Y^i$$

The j^{th} coordinate of this is exactly Z^j.

□

c) Find the explicit formula for the coordinates of $[X, Y]$ using coordinates of X and Y in the standard basis in $\mathbb{R}^n$.
Solution. Since $Z^j = [X, Y]^j$, it follows that

$$[X, Y]^j = \left(\sum_{i=1}^{n} X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}\right)$$

□

Problem 4. Prove that the same statements a) and b) as in the previous problem hold for tangent vector fields X and Y on a k-surface Σ in $\mathbb{R}^n$. Find the analogue of the formula c), but now in local coordinates $u^1, \ldots, u^k$ and the basis $\mathbf{r}_{u^1}, \ldots, \mathbf{r}_{u^k}$. Prove that in this case the commutator $[X, Y]$ is also tangent to Σ.
Solution. Since our solution in Problem 3 applies generally, we see that an analogous formula is

$$[X, Y]^j = \left(\sum_{i=1}^{k} X^i \frac{\partial Y^j}{\partial u^i} - Y^i \frac{\partial X^j}{\partial u^i}\right)$$

To see why $[X, Y]$ is tangent to Σ, notice that

$$\partial_{[X,Y]} r = \sum_{j=1}^{k} [X, Y]^j \frac{\partial r}{\partial u^j} = \sum_{j=1}^{k} [X, Y]^j \cdot \mathbf{r}_{u^j}$$

which means that $[X, Y]$ has a representation using the basis vectors of the tangent space. Thus, it is tangent to Σ.

□

Problem 5. Let M be the two-dimensional torus $\mathbb{T}^2$ defined as a subset of $\mathbb{R}^4$ by equations

$$\left(x^1\right)^2 + \left(x^2\right)^2 = 1, \quad \left(x^3\right)^2 + \left(x^4\right)^2 = 1,$$

and let A be the point $\left(\frac{3}{5}, \frac{4}{5}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Find a parametrization of this torus. Provide a basis f_1, f_2 in $T_A M$ and a basis η_1, η_2 in $N_A M$. Find the orthogonal projector $P_A : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ on

the tangent space T_AM.
Provide a basis e_1, e_2 in tangent vector fields such that $e_i(A) = f_i$. Find $\partial_{e_i} e_j$. Find the orthogonal projector P at an arbitrary point and compute $\nabla_{e_i} e_j$ and $B(e_i, e_j)$ directly by the definitions

$$\nabla_X Y = P(\partial_X Y), \quad B(X,Y) = (Id - P)(\partial_X Y)$$

Solution. Based on the provided equations, we may write

$$r(s,t) = (\cos t, \sin t, \cos s, \sin s)$$

The tangent space T_AM is given by span(r_s, r_t). We want to find $r_t(A)$ and $r_s(A)$, we can do so by noting that $A = \left(\frac{3}{5}, \frac{4}{5}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, so we can find the local coordinates s, t of A through inverse trigonometric functions. So,

$$r_t = (-\sin t, \cos t, 0, 0) \qquad r_s = (0, 0, -\sin t, \cos t)$$
$$r_t(A) = \left(-\frac{4}{5}, \frac{3}{5}, 0, 0\right) \qquad r_s(A) = \left(0, 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

These are basis vectors f_1, f_2 for the tangent vector space T_AM. We may simply consider the opposite reciprocals of the tangent vectors to find the basis vectors of the normal vector space

$$\eta_1 = \left(\frac{3}{5}, \frac{4}{5}, 0, 0\right) \qquad \eta_2 = \left(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Notice that f_1, f_2, η_1, η_2 is an orthonormal basis in $\mathbb{R}^4$. The projector is then given by

$$P_A(\alpha^1 f_1 + \alpha^2 f_2 + \beta^1 \eta_1 + \beta^2 \eta_2) = \alpha^1 f_1 + \alpha^2 f_2$$

To find the normal vectors, we need to solve a system of equations, the solution will have codimension$(\Sigma) = n - k$ as its dimension.
The second part of the problem is to provide a basis. From our previous calculation, we may just choose $e_i(A) = f_i$, so

$$e_1 = r_t = (-\sin t, \cos t, 0, 0) \quad e_2 = r_s = (0, 0, -\sin s, \cos s)$$
$$\eta_1(s,t) = (\cos t, \sin t, 0, 0) \quad \eta_2(s,t) = (0, 0, \cos s, \sin s)$$

To calculate the derivative, consider

$$\partial_{e_1} e_1 = \frac{\partial e_1}{\partial t} = (-\cos t, -\sin t, 0, 0)$$

For the covariant derivative and second fundamental form, we have

$$\nabla_{e_1} e_1 = P_A((-\cos t, -\sin t, 0, 0)) = P_A(-\eta_1) = 0$$
$$B(e_1, e_1) = (\mathrm{Id} - P_A)(-\eta_1) = -\eta_1$$

Calculating the other combinations are similar, we have

$$\partial_{e_1} e_2 = \partial_{e_2} e_1 = \frac{\partial e_1}{\partial s} = 0$$
$$\partial_{e_2} e_2 = \frac{\partial e_2}{\partial s} = (0, 0, -\cos s, -\sin s) = -\eta_2$$

Lastly, the covariant derivative and second fundamental form

$$\nabla_{e_2} e_1 = P_A(0) = 0 \quad \nabla_{e_2} e_2 = P_A(-\eta_2) = 0$$
$$B(e_1, e_2) = (\mathrm{Id} - P_A)(0) = 0 \quad B(e_2, e_2) = (\mathrm{Id} - P_A)(-\eta_2) = -\eta_2$$

□

The Covariant Derivative

Class 6: Covariant Derivative

Recall that if $X, Y, Z \in T_A\Sigma$, then

$$\begin{aligned} \partial_X Y &= P(\partial_X Y) + (\mathrm{Id} - P)(\partial_X Y) \\ &= \nabla_X Y + B(X, Y) \end{aligned}$$

where $\nabla_X Y$ is called the covariant derivative and $B(X, Y)$ is called the second fundamental form. Recall also the properties

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

and

$$\partial_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$$

which says ∇ is a metric connection.
If $e_1, \ldots, e_k$ is a basis in $T_A\Sigma$, then $\nabla_{e_i} e_j \in T_A\Sigma$ and that

$$\nabla_{e_i} e_j = \Gamma_{ij}^p e_p$$

Problem 1. Consider a basis $e_1 = \mathbf{r}_{u^1}, \ldots, e_k = \mathbf{r}_{u^k}$ in tangent vector fields on a k-surface Σ in $\mathbb{R}^n$. Prove that $\Gamma_{ij}^l = \Gamma_{ji}^l$.
Solution. We consider

$$\begin{aligned} \nabla_{e_i} e_j = \nabla_{\mathbf{r}_{u^i}} \mathbf{r}_{u^j} &= P(\partial_{\mathbf{r}_{u^i}} \mathbf{r}_{u^j}) = P\left(\frac{\partial \mathbf{r}_{u^j}}{\partial u^i}\right) \\ &= P\left(\frac{\partial^2 \mathbf{r}}{\partial u^i \partial u^j}\right) = P\left(\frac{\partial^2 \mathbf{r}}{\partial u^j \partial u^i}\right) = \nabla_{e_j} e_i \end{aligned}$$

So, notice that

$$\Gamma_{ij}^p e_p = \nabla_{e_i} e_j = \nabla_{e_j} e_i = \Gamma_{ji}^p e_p$$

Thus, we see that $\Gamma_{ij}^p = \Gamma_{ji}^p$.

□

Remark. Using the basis from before, observe

$$\partial_{\mathbf{r}_{u^i}} \mathbf{r}_{u^j} = \frac{\partial^2 \mathbf{r}}{\partial u^i \partial u^j} = \Gamma_{ij}^1 \frac{\partial \mathbf{r}}{\partial u^1} + \ldots + \Gamma_{ij}^k \frac{\partial \mathbf{r}}{\partial u^k} + B\left(\frac{\partial \mathbf{r}}{\partial u^i}, \frac{\partial \mathbf{r}}{\partial u^j}\right)$$

This follows directly from the definition of $\partial_X Y$. Recall the first Frenet equation that $\dot{v} = kn$, or in a more familiar form $\ddot{r} = 0 \cdot \dot{r} + kn$.

Problem 2. Prove the formula

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2}(\partial_X \langle Y, Z \rangle + \partial_Y \langle Z, X \rangle - \partial_Z \langle X, Y \rangle + \langle Z, [X, Y] \rangle + \langle Y, [Z, X] \rangle - \langle X, [Y, Z] \rangle)$$

Solution. This formula is interesting because it helps us to define the connection (or covariant derivative) on an abstract manifold. To derive this formula, we want to use the fourth and fifth properties that we discussed about the Levi-Civita connection

$$\begin{aligned} 2\text{RHS} &= \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle + \langle \nabla_Y Z, X \rangle + \langle Z, \nabla_Y X \rangle - \langle \nabla_Z X, Y \rangle - \langle X, \nabla_Z Y \rangle \\ & \langle Z, \nabla_X Y - \nabla_Y X \rangle + \langle Y, \nabla_Z X - \nabla_X Z \rangle - \langle X, \nabla_Y Z - \nabla_Z Y \rangle \\ &= 2 \langle \nabla_X Y, Z \rangle \end{aligned}$$

Again, this formula is needed to show that the covariant derivative exists and is unique on any abstract manifold.

□

Problem 3. Consider a basis $e_1 = \mathbf{r}_{u^1} \ldots, e_k = \mathbf{r}_{u^k}$ in tangent vector fields on a k-surface Σ in $\mathbb{R}^n$. Let $g_{ij} = \langle e_i, e_j \rangle$ be metric coefficients, i.e. the matrix elements of the first fundamental form. Using the formula from the previous problem, find how to express the Christoffel symbols if we know only the metric coefficients g_{ij}. How do we do this for an arbitrary basis?

Solution. If we consider vectors in the given basis, notice that $\langle \nabla_X Y, e_j \rangle = (\nabla_X Y)^j$. The problem is asking us to consider $\nabla_{\mathbf{r}_{u^i}} \mathbf{r}_{u^j} = \Gamma^p_{ij} \mathbf{r}_{u^p}$ and to determine $\Gamma^p_{ij}(u^1, \ldots, u^k)$, that is the Christoffel symbol as a function of the local coordinates. Consider the commutator

$$\begin{aligned} [\mathbf{r}_{u^i}, \mathbf{r}_{u^j}] &= \nabla_{\mathbf{r}_{u^i}} \mathbf{r}_{u^j} - \nabla_{\mathbf{r}_{u^j}} \mathbf{r}_{u^i} \\ &= P(\partial_{\mathbf{r}_{u^i}} \mathbf{r}_{u^j} - \partial_{\mathbf{r}_{u^j}} \mathbf{r}_{u^i}) \\ &= P\left(\frac{\partial^2 \mathbf{r}}{\partial u^i u^j} - \frac{\partial^2 \mathbf{r}}{\partial u^j u^i} \right) = 0 \end{aligned}$$

Recall our formula for $\langle \nabla_X Y, Z \rangle$ and substitute $X = \mathbf{r}_{u^i}$, $Y = \mathbf{r}_{u^j}$, and $Z = \mathbf{r}_{u^l}$. Then, it follows from what we have shown above that $[X, Y] = 0$, $[Y, Z] = 0$, and $[X, Z] = 0$. As a result, we are only left with the former summands:

$$\begin{aligned} \left\langle \nabla_{\mathbf{r}_{u^i}} \mathbf{r}_{u^j}, \mathbf{r}_{u^l} \right\rangle &= \frac{1}{2}(\partial_{\mathbf{r}_{u^i}} \langle \mathbf{r}_{u^j}, \mathbf{r}_{u^l} \rangle + \partial_{\mathbf{r}_{u^j}} \langle \mathbf{r}_{u^l}, \mathbf{r}_{u^i} \rangle - \partial_{\mathbf{r}_{u^l}} \langle \mathbf{r}_{u^i}, \mathbf{r}_{u^j} \rangle) \\ &= \frac{1}{2}(\partial_{\mathbf{r}_{u^i}} g_{jl} + \partial_{\mathbf{r}_{u^j}} g_{il} - \partial_{\mathbf{r}_{u^l}} g_{ij}) \\ &= \frac{1}{2}\left(\frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l} + \frac{\partial g_{il}}{\partial u^j} \right) \end{aligned}$$

On the other hand, we also have

$$\left\langle \nabla_{\mathbf{r}_{u^i}} \mathbf{r}_{u^j}, \mathbf{r}_{u^l} \right\rangle = \left\langle \Gamma^p_{ij} \mathbf{r}_{u^p}, \mathbf{r}_{u^l} \right\rangle = \Gamma^p_{ij} \langle \mathbf{r}_{u^p}, \mathbf{r}_{u^l} \rangle = \Gamma^p_{ij} g_{pl}$$

Hence, combining these equalities, we have

$$\Gamma^p_{ij} g_{pl} = \frac{1}{2}\left(\frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l} + \frac{\partial g_{il}}{\partial u^j}\right)$$

To extract the Christoffel symbol, we must multiply by the inverse matrix. We denote the inverse matrix with upper indices, meaning $g_{pl}g^{lk} = \delta^k_p$, where δ indicates the Kronecker delta. So, we have

$$\Gamma^p_{ij} g_{pl} \cdot g^{lk} = \frac{1}{2} g^{lk}\left(\frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l} + \frac{\partial g_{il}}{\partial u^j}\right)$$
$$\Gamma^p_{ij}\delta^k_p = \frac{1}{2} g^{kl}\left(\frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l} + \frac{\partial g_{il}}{\partial u^j}\right)$$
$$\Gamma^p_{ij} = \frac{1}{2} g^{pl}\left(\frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l} + \frac{\partial g_{il}}{\partial u^j}\right)$$

To do this in an arbitrary basis, consider the formula from Problem 2. To be explicit, if $e_1, \dots, e_k$ is an orthonormal basis, then $\langle e_i, e_j \rangle = 0$. May need more details.

□

Problem 4. Consider a unit sphere

$$\mathbf{r}(\varphi, \psi) = (\cos\varphi \sin\psi, \sin\varphi \sin\psi, \cos\psi).$$

Consider a basis $e_1 = \mathbf{r}_\varphi, e_2 = \mathbf{r}_\psi$ in tangent vector fields. Find the Christoffel symbols using their definition. Find the Christoffel symbols starting from the first fundamental form. Compare the answers.

Solution. With the basis r_φ, r_ψ, we have

$$\nabla_{r_\varphi} r_\varphi = \Gamma^1_{11} r_\varphi + \Gamma^2_{11} r_\psi$$

using the formula we derived in the previous problem. Notice also that

$$\frac{\partial^2 r}{\partial \varphi^2} = \Gamma^1_{11} r_\varphi + \Gamma^2_{11} r_\psi + b_{ii} m$$

and $\Gamma^1_{11} = \frac{\langle r_{\varphi\varphi}, r_\varphi \rangle}{\langle r_\varphi, r_\varphi \rangle}$, which we may think of as a projection. We compute

$$r_\varphi = (-\sin\varphi \sin\psi, \cos\varphi \sin\psi, 0)$$
$$r_\psi = (\cos\varphi \cos\psi, \sin\varphi \cos\psi, -\sin\psi)$$
$$r_{\varphi\varphi} = (-\cos\varphi \sin\psi, -\sin\varphi \sin\psi, 0)$$

So,

$$\Gamma^1_{11} = \frac{\cos\varphi \sin\varphi \sin^2\psi - \cos\varphi \sin\varphi \sin^2\psi}{\dots} = 0$$
$$\Gamma^2_{11} = \frac{-\cos^2\varphi \cos\psi \sin\psi - \sin^2\varphi \cos\psi \sin\psi}{1} = -\cos\psi \sin\psi$$

Now, we want to apply the formula to see if we get the exact same result

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(\frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l} + \frac{\partial g_{il}}{\partial u^j}\right)$$

Recall that the first fundamental form of the sphere is $g = \begin{pmatrix} \sin^2\psi & 0 \\ 0 & 1 \end{pmatrix}$ and the inverse matrix just inverts entries along the diagonal of g. We compute

$$\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2} g^{11} \left(\frac{\partial g_{11}}{\partial u^1} - \frac{\partial g_{11}}{\partial u^1} + \frac{\partial g_{11}}{\partial u^1} \right) + \frac{1}{2} g^{12}(\dots) = 0 \\
\Gamma^2_{11} &= \frac{1}{2} g^{21}(\dots) + \frac{1}{2} g^{22} \left(\frac{\partial g_{12}}{\partial u^1} - \frac{\partial g_{11}}{\partial u^2} + \frac{\partial g_{21}}{\partial u^1} \right) \\
&= -\frac{1}{2} \cdot 2\cos\psi \sin\psi = -\cos\psi\sin\psi
\end{aligned}$$

The other calculations are similar.

□

Problem 5. Find the Weingarten operator W_m for the sphere $x^2 + y^2 + z^2 = R^2$, where m is a unitary normal vector. Can you find the answer without using a basis in the tangent vector fields?

Solution. For 2-dimensional surfaces in $\mathbb{R}^3$, in the basis r_{u^1}, r_{u^2}, the Weingarten operator is given by

$$W_m = \mathrm{I}^{-1}\mathrm{II}$$

So we may directly compute

$$W_m = \begin{pmatrix} \sin^2\psi & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\sin^2\psi & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

To derive the result without using the basis, we may consider normal sections of the sphere. A normal section of the sphere is a great circle. Assuming our radius is 1, the curvature of every normal section is 1, this means that in the basis of principal directions, we have $W = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Since W is the identity, under a change of basis it remains the same (and it is the same in the basis of the Weingarten operator). The signs are different because of our choice of normal vector.

□

Class 7: Parallel Transport

The aspects of geometry that depend only on the first fundamental form are called the intrinsic properties. Parallel transport is an intrinsic property. By definition, a vector field $Y(t)$ is parallel along γ if any of the following are satisfied

1. $$P_{T_A\Sigma}\left(\frac{dY}{dt}\right) = 0$$

2. $$\nabla_{\dot{\gamma}} Y = 0$$

3.
$$\frac{dY^j}{dt} + \Gamma^j_{ip} Y^i \dot{u}^p = 0$$

Here, we mean that $Y^1, \ldots, Y^k$ are local coordinates of Y and $u^1 \ldots, u^k$ are local coordinates of our curve γ.

Problem 1. Find the Christoffel symbols for a circular cylinder in $\mathbb{R}^3$ in suitable coordinates. Find the result of parallel transport on a circular cylinder by solving ODEs. Explain how it depends on a curve.

Solution. Consider the parametrization of the cylinder

$$r(\varphi, h) = (R\cos\varphi, R\sin\varphi, h)$$

Taking the partial derivatives, we have

$$\begin{aligned} r_\varphi &= (-R\sin\varphi, R\cos\varphi, 0) \\ r_h &= (0, 0, 1) \end{aligned}$$

So, the first fundamental form is

$$\mathrm{I} = \begin{pmatrix} R^2 & 0 \\ 0 & 1 \end{pmatrix}$$

Recalling the formula for the Christoffel symbols we have

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{lj}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l} + \frac{\partial g_{il}}{\partial u^j} \right)$$

Since the entries of our first fundamental form are constant, we know that $\Gamma^k_{ij} = 0$. In order to satisfy parallel transport, we must have

$$\dot{Y}^j + \Gamma^j_{ip} Y^i \dot{u}^p$$

for $j = 1, 2$. Since $\Gamma^j_{ip} = 0$ we simply have the system

$$\begin{cases} \dot{Y}^1 = 0 \\ \dot{Y}^2 = 0 \end{cases}$$

The solutions are just $Y^1(t) = Y^1_0$ and $Y^2(t) = Y^2_0$ for some constants Y^1_0 and Y^2_0 which are determined by the initial conditions. So $Y(t) = Y_0$ in local coordinates, this just means the vector is constant. Note that vectors with the same local coordinates may remain the same in global coordinates, as in the case of r_h, or they may change in global coordinates, as in the case of r_φ.

□

Problem 2. Consider a two-dimensional surface Σ in $\mathbb{R}^3$. Prove that a vector field $Y(t)$ on a curve $\gamma(t)$ is parallel along γ if and only if for all t one has

$$\frac{dY}{dt} \perp T_{\gamma(t)}\Sigma$$

Solve now again Problem 1 using this result, without ODEs.
Solution. Using the first definition of parallel, since Y is parallel, we have

$$P_{T_A\Sigma}(\dot{Y}) = 0$$

As a result, we see that our vector $\dot{Y}$ is parallel to the unit normal m (it has component in the tangent plane). As a result, we see that $\dot{Y} \perp T_{\gamma(t)}\Sigma$.
An alternative way to solve Problem 1 is to consider $Y(t) = (0, 0, 1)$. Since $\dot{Y} = (0, 0, 0)$, it follows that $\dot{Y} \parallel m$, so it is a parallel vector field. In order to determine r_φ, we can measure its angle α to r_h. Since parallel transport preserves angles, we can then consider r_h at a new point, say B, and measure α in the same direction to find r_φ

□

Problem 3. Find the Christoffel symbols for a unit sphere. Solving the corresponding ODE system, find the angle of rotation of a tangent vector to a 2-dimensional sphere after parallel transport along a parallel $\theta = \theta_0$ by angle α.
Solution. Consider the parametrization of the sphere

$$r(\varphi, \theta) = (\cos\varphi\cos\theta, \sin\varphi\cos\theta, \sin\theta)$$

The curve is parametrized as $\gamma(t) = (\varphi(t), \theta(t))$ in local coordinates. If we are considering one of the parallels of the sphere, then we have the parametrization $\gamma(t) = (t, \theta_0)$ in particular. From our previous class, we know that the Christoffel symbols are

$$\Gamma^1_{11} = 0 \quad \Gamma^1_{12} = -\tan(\theta) \quad \Gamma^1_{22} = 0$$
$$\Gamma^2_{11} = \cos\theta\sin\theta \quad \Gamma^2_{12} = 0 \quad \Gamma^2_{22} = 0$$

We have the differential equations

$$\begin{cases} \dot{Y}^1 + \Gamma^1_{12}Y^1\dot{u}^2 + \Gamma^1_{21}Y^2\dot{u}^1 = 0 \\ \dot{Y}^2 + \Gamma^2_{11}Y^1\dot{u}^1 = 0 \end{cases}$$

Substituting in for our Christoffel symbols, we have

$$\dot{Y}^1 - \tan\theta(Y^1\dot{u}^2 + Y^2\dot{u}^1) = 0$$
$$\dot{Y}^2 + \cos\theta\sin\theta Y^1\dot{u}^1 = 0$$

Since $\gamma(t) = (t, \theta_0)$, the velocity vector in local coordinates is $(\dot{u}^1, \dot{u}^2) = (1, 0)$. As a result, our equations become

$$\dot{Y}^1 - \tan(\theta_0)Y^2 = 0$$
$$\dot{Y}^2 + \cos\theta_0\sin\theta_0 Y^1 = 0$$

These are linear differential differential equations with constant coefficients, so this may be solved with regular methods in differential equations. However, we may also notice that

$$\ddot{Y}^1 - \tan\theta_0\dot{Y}^2 = 0$$
$$\ddot{Y}^1 = -\tan\theta_0\cos\theta_0\sin\theta_0 Y^1$$
$$\ddot{Y}^1 = -\sin^2\theta_0 Y^1$$

The last equation is one of the standard differential equations, we know that solution is

$$Y^1 = A\cos(\sin\theta_0 t) + B\sin(\sin\theta_0 t)$$
$$Y^2 = -A\cos\theta_0\sin(\sin\theta_0 t) + B\cos\theta_0\cos(\sin\theta_0 t)$$

where we have computed the second equation from our knowledge that $\dot{Y}^1 - \tan(\theta_0)Y^2 = 0$. Suppose we have $Y_0 = (0,1)$ in the coordinates r_φ, r_θ and we want to find the solution of the Cauchy problem with this initial data. With this initial data, we see that

$$Y^1(0) = A = 0$$
$$Y^2(0) = B\cos\theta_0 = 1$$

So making substitutions in for A and B, we see that the result of the parallel transport is

$$Y^1 = \frac{\sin(\sin\theta_0 t)}{\cos\theta_0} \quad Y^2 = \cos(\sin\theta_0 t)$$

Now, we consider the case of parallel transport of this vector, $Y_0 = r_\theta$ after moving through $\alpha = 2\pi$. To find the result, we need to simply substitute $t = 2\pi$ into the formula for Y^1, Y^2 above. We get $Y^1 = \frac{\sin(\sin\theta_0 2\pi)}{\cos\theta_0}$ and $Y^2 = \cos(\sin\theta_0 2\pi)$. To find the angle between this vector and $(0,1)$ we need to take the scalar product, which involves using the first fundamental form. First, notice that

$$\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}\cos^2\theta_0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = 1$$

which shows us that the length of $(0,1)$ is 1. Since parallel transport preserves lengths, we know that the length of the new vector will also be 1. Thus we do not need to normalize and

$$\cos\beta = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}\cos^2\theta_0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\dots\\ \cos(\sin\theta_0 2\pi)\end{pmatrix}$$

This shows us that $\beta = 2\pi\sin\theta_0$.

□

Now, we move to some equivalent definitions of a **local isometry.** Let $\Sigma_1, \Sigma_2 \subseteq \mathbb{R}^3$ be surfaces and let $r : \Sigma_1 \to \Sigma_2$ is a map. Then, r is a **local isometry** if it satisfies any of the following

1. For all $A \in \Sigma_1$
$$I_A = I_{r(A)}$$
that is, under the map r, the first fundamental form at $A \in \Sigma_1$ is equal to the first fundamental form at its image $r(A) \in \Sigma_2$.

2. For all curves $\gamma \subseteq \Sigma_1$ and $r(\Sigma) \subseteq \Sigma_2$, we have
$$\ell(\gamma) = \ell(r(\gamma))$$
The length of the curve on the first surface is equal to the length of the image of that curve on the second surface.

Intrinsic geometry is the geometry that stays invariant under local isometries.

Problem 4. Describe parallel transport along a loop on a circular cone in $\mathbb{R}^3$. Explain how it depends on a curve.

Solution. Our idea is to create an isometry between the circular cone and the plane. First, let us consider a plane in polar coordinates

$$\tilde{r}(\varphi, h) = (h\cos\varphi, h\sin\varphi, 0)$$

Taking the derivatives, we have

$$r_h = (\cos\varphi, \sin\varphi, 0) \qquad r_\varphi = (-h\sin\varphi, h\cos\varphi, 0)$$

The first fundamental form is

$$\mathrm{I}(\varphi, h) = \begin{pmatrix} h^2 & 0 \\ 0 & 1 \end{pmatrix}$$

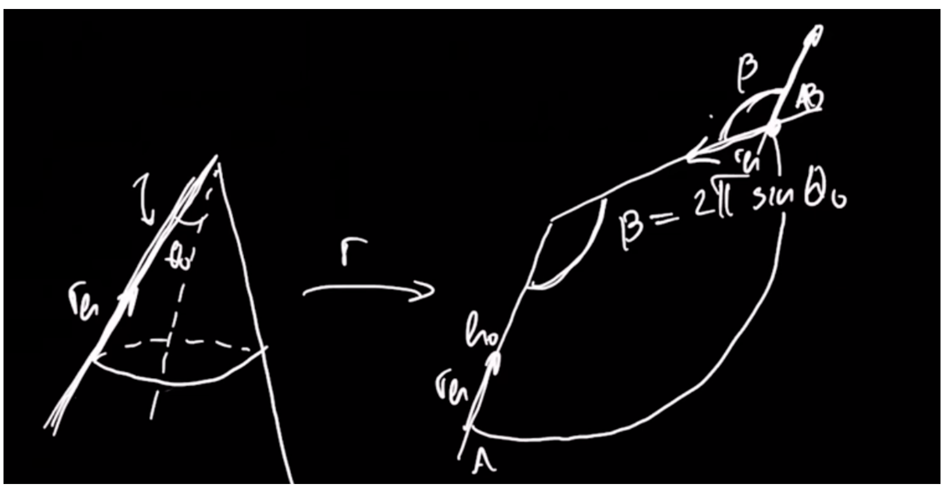
Now, we consider the parametrization of the cone

$$r(\varphi, h) = \left(h\sin\theta_0 \cos\left(\frac{\varphi}{\sin\theta_0}\right), h\sin\theta_0 \sin\left(\frac{\varphi}{\sin\theta_0}\right), -h\cos\theta_0 \right)$$

where θ_0 is the angle between the altitude of the cone and its edge. Calculating the first fundamental form, we have

$$\mathrm{I} = \begin{pmatrix} h^2 & 0 \\ 0 & 1 \end{pmatrix}$$

So, there is an isometry, based on the first definition. Considering this isometry, we see that

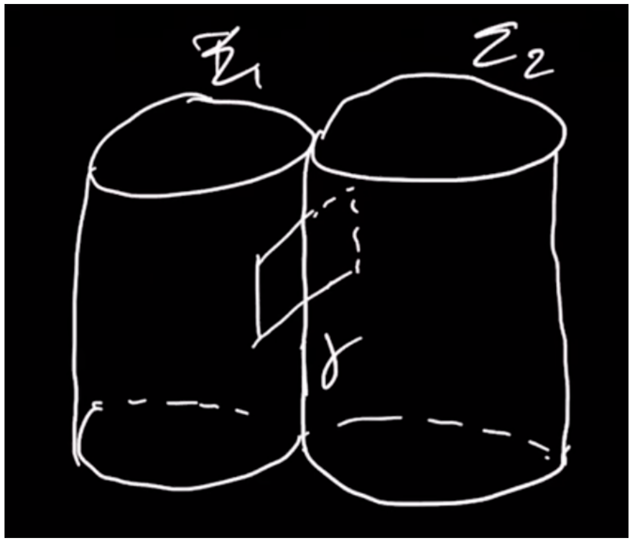


The result of the parallel transport involves the vector turning by the angle $\beta = 2\pi\sin\theta_0$. This explanation is not fully rigorous.

□

Problem 5. Consider two surfaces such that both of them contain the same curve and their tangent planes in the points of this curve coincide. Prove that the result of the parallel transport of a tangent vector along this curve on both surfaces is the same. Find now the answer of problem 3 without solving ODEs.

The usual example to illustrate this principle simply are two cylinders which are placed next to each other such that the tangent plane of one of the circular curves touches both cylinders, shown below



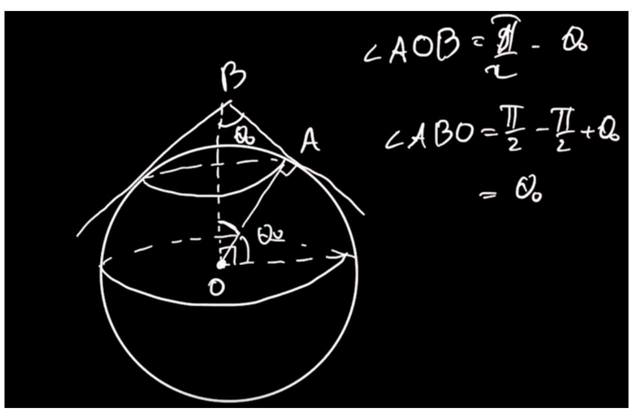
Solution. Suppose that $Y(t)$ is a vector field parallel along the curve on Σ_1. Then, by definition, we have

$$P_{T_A\Sigma_1}(\dot{Y}) = 0$$

But from the premise, we know that the tangent planes of the curves coincide, and as a result we see $P_{T_A\Sigma_2}(\dot{Y}) = P_{T_A\Sigma_1}(\dot{Y})$ and conclude that

$$P_{T_A\Sigma_2}(\dot{Y}) = 0$$

Thus, $Y(t)$ is parallel along γ on Σ_2. Since $Y(t)$ is parallel on Σ_2, it describes the parallel transport of the tangent vector on the second surface as well. The idea mentioned in this problem is related to why Problems 3 and 4 produced the same answer. Consider the following setup.



We may choose a tangent vector on the cone and observe from our proof in Problem 4 that it turns by $2\pi \sin\theta_0$. Using the fact that we have the same curve on the circle and cone and that their tangent planes at the points of this curve coincide, we conclude that transport along the circle has the same result, solving Problem 3 without ODEs.

□

Class 8: Geodesics, Push Forward and Isometries

Differential Forms

Class 9: Differential Forms 1

Let V be a vector space generated by the basis $e_1, \ldots, e_n$. Let V^* be a dual space, where $\xi \in V^*$ and $\xi : V \to \mathbb{R}$ is generated by the dual basis $e^1, \ldots, e^n$. The property that defines the dual basis is that it is composed of the coordinate functionals

$$e^i(e_j) = \delta^i_j$$

where δ^i_j is the Kronecker delta. To determine a linear function, it is enough to determine how it behaves on basis vectors of the domain, since we may extend by linearity. The elements ξ are called covectors.

We also have discussed k-forms in lecture, which are skew-symmetric, polylinear functions. Our notation for a generic k-form is $\omega \in \wedge^k V^*$ and $\omega : V^k \to \mathbb{R}$. Here, the skew-symmetric condition implies the following equalities

$$\omega(v_1, v_2, \ldots, v_k) = -\omega(v_2, v_1, \ldots, v_k)$$
$$\omega(v_{\pi(1)}, \ldots, v_{\pi(k)} = \text{sgn}(\pi) \cdot \omega(v_1, v_2, \ldots, v_k)$$

We note that these are not differential forms until we specify both a *surface* and the *vector space*, usually the tangent plane at a point. Forms are important for describing connections between algebraic and differential properties of manifolds, and cohomology of manifold.

If we have two forms $\tau \in \wedge^k V^*$ and $\sigma \in \wedge^l V^*$, then we can take the wedge product to produce a new form, $\tau \wedge \sigma \in \wedge^{k+l} V^*$, where

$$(\tau \wedge \sigma)(v_1, \ldots, v_{k+l}) = \frac{1}{(k+l)!} \sum_{\pi \in S_{k+l}} \text{sgn}(\pi) \cdot \tau(v_{\pi(1)}, \ldots, v_{\pi(k)}) \cdot \sigma(v_{\pi(k+1)}, \ldots, v_{\pi(k+l)})$$

Problem 1. Consider covectors $\omega_1, \ldots, \omega_k$ and vectors $X_1, \ldots, X_k$. Prove that

$$\omega_1 \wedge \ldots \wedge \omega_k (X_1, \ldots, X_k) = \frac{1}{k!} \begin{vmatrix} \omega_1(X_1) & \ldots & \omega_1(X_k) \\ \vdots & \ddots & \vdots \\ \omega_k(X_1) & \ldots & \omega_k(X_k) \end{vmatrix}.$$

Solution. We proceed by induction on k. The base case is $n = 1$. It is clear that

$$\omega_1(X_1) = \frac{1}{1!} \cdot \det(\omega_1(X_1))$$

Now, suppose that the formula holds for some $k-1$. Then

$$\begin{aligned}
&(w_1 \wedge \ldots \wedge w_{k-1}) \wedge w_k(X_1, \ldots, X_k) \\
&= \frac{1}{k!} \sum_{\pi \in S_k} \operatorname{sgn}(\pi) \cdot \omega_1 \wedge \ldots \wedge \omega_{k-1}(X_{\pi(1)}, \ldots, X_{\pi(k-1)}) \cdot \omega_k(X_{\pi(k)}) \\
&= \frac{(k-1)!}{k!} \sum_{i=1}^{k} (-1)^{k-i} \omega_1 \wedge \ldots \wedge \omega_{k-1}(X_1, \ldots, \hat{X}_i, \ldots, X_k)\, \omega_k(X_i) \\
&= \frac{1}{k} \sum_{i=1}^{k} (-1)^{k-i} \omega_1 \wedge \ldots \wedge \omega_{k-1}(X_1, \ldots, \hat{X}_i, \ldots, X_k)\, \omega_k(X_i)
\end{aligned}$$

Note that $\widehat{X_i}$ is shorthand notation, where

$$(X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n) = (X_1, X_2, \ldots, \widehat{X_i}, \ldots, X_n)$$

In the last equality, there were a number of steps done at once. First, we reordered the arguments. We set $\pi(k) = i$ and make the following permutation

$$\begin{pmatrix} X_1 & \ldots & X_i & \ldots & X_{k-1} & X_k \\ X_1 & \ldots & \widehat{X_i} & \ldots & X_k & X_i \end{pmatrix} \circ \begin{pmatrix} X_{\pi(1)} & \ldots & X_{\pi(i)} & \ldots & X_{\pi(k)} \\ X_1 & \ldots & X_i & \ldots & X_k \end{pmatrix}$$

We permute the initial order to that of the ascending order. The value of (-1) raised to the number of transpositions made in the k-form to achieve this new order is equal to $\operatorname{sgn}(\pi)$ by skew-symmetry. Once we have the indices in ascending order, we must make $k-i$ transpositions to move the i^{th} index to the last place.

Since $k-i$ transpositions are made, the sign of this permutation is $(-1)^{k-i}$. We can do this because the original definition of the wedge product has all the permutations already, we are simply trying to determine the sign in front of a specific permutation based on the ordering we are given. Finally, the term $(k-1)!$ comes from the fact that there are $(k-1)!$ terms with the same multiplicand $\omega_k(X_i)$. By our induction assumption, we have

$$\omega_1 \wedge \ldots \wedge \omega_{k-1}(X_1, \ldots, \hat{X}_i, \ldots, X_k) = \frac{1}{(k-1)!} \begin{vmatrix} \omega_1(X_1) & \ldots & \widehat{\omega_i(X_i)} & \ldots \omega_1(X_k) \\ \vdots & & & \\ \omega_{k-1}(X_1) & \ldots & \widehat{\omega_{k-1}(X_i)} & \ldots \omega_{k-1}(X_k) \end{vmatrix}$$

However, now we see that

$$\begin{aligned}
\frac{1}{k!} \begin{vmatrix} \omega_1(X_1) & \ldots & \omega_1(X_k) \\ \vdots & \ddots & \vdots \\ \omega_k(X_1) & \ldots & \omega_k(X_k) \end{vmatrix} &= \frac{1}{k!} \sum_{i=1}^{k} (-1)^{k+i} \omega_k(X_i) \det(\hat{x_i}) \\
&= \frac{(k-1)!}{k!} \sum_{i=1}^{k} (-1)^{k-i} \omega_k(X_i)\, \omega_1 \wedge \ldots \wedge \omega_{k-1}(X_1, \ldots, \hat{X}_i, \ldots, X_k) \\
&= (w_1 \wedge \ldots \wedge w_{k-1}) \wedge w_k(X_1, \ldots, X_k)
\end{aligned}$$

□

Problem 2. Prove that if vectors v_1, v_2 are linearly dependent then for any form ω one has

$$\omega(v_1, v_2) = 0.$$

Solution 1. Given two 1-forms such that $\omega = e^{i_1} \wedge e^{i_2}$ and linearly dependent vectors $v_1 = \alpha v_2$, we see that

$$\begin{aligned} e^{i_1} \wedge e^{i_2}(v_1, v_2) &= e^{i_1} \wedge e^{i_2}(v_1, \alpha v_1) = \alpha e^{i_1} \wedge e^{i_2}(v_1, v_1) \\ &= \frac{\alpha}{2} \begin{vmatrix} e^{i_1}(v_1) & e^{i_1}(v_1) \\ e^{i_2}(v_1) & e^{i_2}(v_1) \end{vmatrix} = 0 \end{aligned}$$

□

Solution 2. We may simply use the definition as well.

$$\omega(v_1, \alpha v_1) = \alpha\omega(v_1, v_1) = -\alpha\omega(v_1, v_1)$$

Thus, $\omega(v_1, v_1) = 0$ by skew symmetry, which means that $\omega(v_1, v_2) = 0$ as well.

□

Problem 3. Prove that if differential 1-forms $\omega_1, \ldots, \omega_p$ are linearly dependent then $\omega_1 \wedge \ldots \wedge \omega_p = 0$. Is the inverse statement true? What can we say about product forms of arbitrary degree?

Solution. By the properties of the wedge product, we have

$$\omega \wedge \omega = (-1)^{\operatorname{sgn}(\omega)\operatorname{sgn}(\omega)} \omega \wedge \omega = (-1)\, \omega \wedge \omega$$

This implies that for any 1-form ω, we have $\omega \wedge \omega = 0$.
Suppose $\omega_1, \ldots, \omega_p$ are linearly dependent. Without loss of generality, we may write

$$\omega_p = \lambda_1\omega_1 + \ldots + \lambda_{p-1}\omega_{p-1}$$

Then, by substitution, we have

$$\begin{aligned} \omega_1 \wedge \ldots \wedge \omega_{p-1} \wedge \omega_p &= \omega_1 \wedge \ldots \wedge \omega_{p-1} \wedge (\lambda_1\omega_1 + \ldots + \lambda_{p-1}\omega_{p-1}) \\ &= \sum_{i=1}^{p-1} \omega_1 \wedge \ldots \wedge \omega_{p-1} \wedge \lambda_i\omega_i \\ &= 0 \end{aligned}$$

We are able to exchange ω_i freely to any position in the form based on the properties of the wedge product, which allows us to produce a 0 in each of the wedge products.

Regarding the inverse statement for 1-forms, it is true. Let $\omega_1, \ldots \omega_p$ be linearly independent covectors. Denote $e^1 = \omega_1, \ldots, e^p = \omega_p$, these are elements of some dual basis V^*. From linear algebra, we know that a set of linearly independent vectors may be extended to a basis for the space, so let

$$e^1, \ldots, e^p, e^{p+1}, \ldots, e^n$$

be a basis for V^*. Now, consider a basis in k-forms in $\wedge^k V^*$

$$e^{i_1} \wedge \ldots \wedge e^{i_k}, \quad i_1 < \ldots < i_k$$

we need the restriction $i_1 < \ldots < i_k$ for linear independence. Since $e^1 \wedge \ldots \wedge e^p$ is an element of this basis, it must be a nonzero element, as the basis is linearly independent. This proves the reverse direction.

The reverse statement for product forms of an arbitrary degree is not true. Suppose $V = \mathbb{R}^3$ with the basis elements e_1, e_2, e_3. Then the basis elements of the space of 2-forms are $\omega_1 = e^1 \wedge e^2$, $\omega_2 = e^2 \wedge e^3$, and $\omega_3 = e^1 \wedge e^3$. It follows that ω_1 and ω_2 are linearly independent and $\omega_1 \wedge \omega_2 = 0$.

$\square$

Problem 4. Prove that there exists a unique operation d called an exterior differential such that

1. for any integer $k \geqslant 0$ it is a map $d : \Omega^k(\Sigma) \longrightarrow \Omega^{k+1}(\Sigma)$ and for $k = 0$ it coincides with the differential of a function,
2. $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$
3. $d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^{\deg \omega} \omega \wedge d\tau$
4. $d^2 = 0$, i.e. for any ω one has $d(d\omega) = 0$

Solution. We begin with the existence and uniqueness of the operation. As we mentioned earlier, a differential is defined relative to some surface and the tangent spaces at points on the surface. So, let $x^1, \ldots, x^n$ be local coordinates for some surface Σ. We have as bases of each space

$$T_A\Sigma = V = \langle r_{x^1}, \ldots, r_{x^n} \rangle \qquad V^* = \left\langle dx^1, \ldots, dx^n \right\rangle$$

Then, we may write a k-form ω as

$$\omega = \sum_{i_1 < \ldots < i_k} w_{i_1 \ldots i_k} \, dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

Taking the differential, which is defined as the operation of the directional derivative, we have

$$d\omega = d \sum = \sum_{i_1 < \ldots < i_k} d(w_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k})$$

Since a function is a zero form, we may write

$$\begin{aligned}
\omega &= \sum_{i_1 < \ldots < i_k} w_{i_1 \ldots i_k} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} \\
d\omega &= \sum_{i_1 < \ldots < i_k} d(w_{i_1 \ldots i_k} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}) \\
&= \sum_{i_1 < \ldots < i_k} dw_{i_1 \ldots i_k} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} \\
&= \sum_{i_1 < \ldots < i_k} \frac{\partial w_{i_1 \ldots i_k}}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}
\end{aligned}$$

Note that we have used Einstein summation for the first term in the wedge product. The formula for the differential of ω that we have obtained above shows that d maps ω to a

$(k+1)$-form in coordinates. Since forms are uniquely determined by their coordinates, d is unique.

To show existence, we define the differential of a k-form using the formula we just derived, and verify the corresponding properties. To be explicit, we define

$$d\omega := \sum_{i_1<\ldots<i_k} \frac{\partial w_{i_1\ldots i_k}}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

If $k=0$, we have

$$df = \frac{\partial f}{\partial x^i} dx^i$$

showing that this definition coincides with the differential of a function. Regarding the second property, we have

$$\begin{aligned} d(\omega_1+\omega_2) &= \sum_{i_1<\ldots<i_k} \frac{\partial(\omega_{1,i_1\ldots i_k}+\omega_{2,i_1\ldots i_k})}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} \\ &= \sum_{i_1<\ldots<i_k} \left(\frac{\partial \omega_{1,i_1\ldots i_k}}{\partial x^i} + \frac{\partial \omega_{2,i_1\ldots i_k}}{\partial x^i}\right) dx^i \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} \\ &= d\omega_1 + d\omega_2 \end{aligned}$$

Now, we show that $d^2=0$. Let f be a 0-form. Observe that

$$\begin{aligned} d^2 f &= d\left(\frac{\partial f}{\partial x^i} \wedge dx^i\right) = d\left(\frac{\partial f}{\partial x^i}\right) \wedge dx^i + \frac{\partial f}{\partial x^i} \wedge d^2 x^i \\ &= \frac{\partial^2 f}{\partial x^j x^i} dx^j \wedge dx^i + \frac{\partial f}{\partial x^i} \wedge d^2 x^i \end{aligned}$$

From the skew-symmetry of the wedge product, we know that if $i \neq j$, then

$$\frac{\partial^2 f}{\partial x^j x^i} dx^j \wedge dx^i + \frac{\partial^2 f}{\partial x^i x^j} dx^i \wedge dx^j = \frac{\partial^2 f}{\partial x^i x^j}(dx^i \wedge dx^j - dx^i \wedge dx^j) = 0$$

In the case where $i=j$, then $dx^i \wedge dx^j = dx^i \wedge dx^i = 0$.
Regarding the second term, we have

$$d^2 x^i = d(dx^i) = \frac{\partial(dx^i)}{\partial x^j} dx^j = 0 \cdot dx^j$$

Next, an inductive argument may be used to show that we obtain 0 for $k>0$. Observe that by Property 3,

$$\begin{aligned} d(dx^{i_1} \wedge \ldots \wedge dx^{i_k}) &= d(dx^{i_1}) \wedge (dx^{i_2} \wedge \ldots \wedge dx^{i_k}) + (-1)^{\deg dx^{i_1}} dx^{i_1} \wedge d(dx^{i_2} \wedge \ldots \wedge dx^{i_k}) \\ &= 0 \end{aligned}$$

□

Problem 5. Find the exterior differential of

$$\omega = \frac{xdy - ydx}{x^2+y^2} \in \Omega^1\left(\mathbb{R}^3 \backslash \{O\}\right).$$

Class 10: Differential Forms 2

Class 11: Differential Forms 3