Differential Geometry Notes

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February 2024

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1 Curves

There are some assumptions we make throughout the course.

1. Smooth refers to the class of $\mathscr{C}^\infty$ functions, that is, the functions that are infinitely differentiable. In other courses, there may be more attention on exactly which class of $\mathscr{C}^k$ functions satisfy specific theorems.

2. We work with an **affine Euclidean space**.

   - An affine space consists of vectors and points, it is a pair (V, X) where V is a vector space, X is some set of points, and $p + v \in X$ for all $p \in X$ and $v \in V$.
   - For any two points $B, C \in X$, there exists a unique v such that $B + v = C$. We may also write
   $$v = \overrightarrow{BC} = C - B$$
   - Given three points $A, B, C \in X$, the triangle relation among the vectors is satisfied:
   $$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$
   This holds as an axiom.
   - Why is the distinction between points and vectors not often emphasized? Suppose we have an affine space and we fix some point O in the space. Then there is a one-to-one correspondence between a point B and a vector $\overrightarrow{OB}$.
   - It is important that the spaces are homogeneous. The intuitive idea of homogeneity is that at every point in the space looks identical. Affine spaces are homogeneous.
   - Let $e_1, \ldots, e_n$ be a basis for V. Then $v \leftrightarrow (v_1, \ldots, v_n)$. Also,
   $$B \leftrightarrow \overrightarrow{OB} \leftrightarrow (\overrightarrow{OB}_1, \ldots, \overrightarrow{OB}_n)$$
   Thus, everything in an affine space may be identified with $\mathbb{R}^n$. Implicit in these above identifications are the choice of a fixed point O and the choice of a basis, respectively.
   - A **frame** in an affine space consists of the choice of some point O together with a basis $e_1, \ldots, e_n$.
   - A **Euclidean space** is a space equipped with a positive definite scalar product (dot product). To simplify our scalar product, it is convenient to choose an orthonormal basis. An orthonormal basis is a basis satisfying $|e_i| = 1$ and $e_i \perp e_j$ for all $i \neq j$. For a choice of orthonormal basis, we have
   $$\langle v, w \rangle = \sum_{i=1}^{n} v_i w_i$$

Definition.

A regular, smooth curve in $\mathbb{R}^n$ is a smooth map $r : [a, b] \to \mathbb{R}^n$ such that for all $t \in [a, b]$, we have

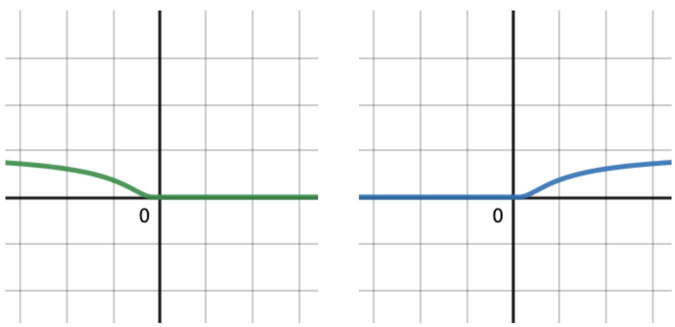
$$\dot{r}(t) \neq 0$$

The nonvanishing derivative is the regularity condition.

Aside. There are two interpretations of curves in geometry. The first is a parametric interpretation (common in differential geometry). The second way is to describe the curve implicitly (common in algebraic geometry), as the solution to some system of homogeneous equations.

1.1 Why is regularity of the curve important?

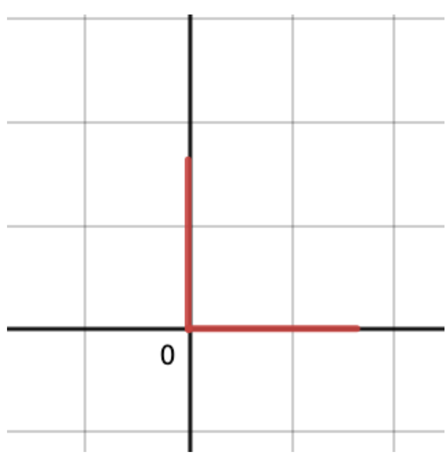
Suppose we have a planar curve $r(t) = (x(t), y(t))$. Let the graphs of the yt-plane and xt-plane be given by:



So, we have

$$x(t) = \begin{cases} 0, & t \leq 0 \\ e^{-\frac{1}{t}}, & t > 0 \end{cases}$$

and $y(t) = x(-t)$. Then, the graph of $r(t)$ appears as an L-shape, as shown below.



What is the problem with this curve? Notice that y is identically 0 for $t > 0$, so its derivative $\dot{y}$ is also 0 for these values of t. By the continuity of the derivative, we have that $\dot{y}(0) = 0$. The same argument over negative values of t for $x(t)$ shows that $\dot{x}(0) = 0$. Thus, $\dot{r}(0) = 0$. We see that this problematic point at the "corner" of $r(t)$ is exactly where the velocity vector vanishes. To be explicit, without the regularity condition, we can have a map with smooth components that is not geometrically smooth. The curve should have a well-defined, nonvanishing tangent vector at every point.

Remark. If $\dot{r}(t) \neq 0$, then at least one of the components of $\dot{r}(t)$ is nonzero, say $\dot{x}(t) \neq 0$. Then, by the inverse function theorem, there exists a smooth function $t = t(x)$. We may plug this function into y to obtain y as a function of x, as $y(t(x))$. Further, $y(t(x))$ is a smooth function since the composition of smooth curves is smooth. This shows that the previous situation where a corner exists is not possible for regular curves.

1.2 How to invent curvature?

We all have an intuitive sense of how curved something appears to be or feels, such as in the picture below:

There are several approaches at how we may define curvature. We first consider a physical point of view, that is, as trajectories or parametrizations. Imagine we are driving along a road and making a sharp turn. We feel a greater force pushing us when making a sharper turn. Recall Newton's second law

$$\vec{F} = m\vec{a} = m\ddot{r}$$

We might think to ignore mass, as this is simply a constant, and conclude that $k = |\ddot{r}|$. However, there is an issue here, as driving the same road in a bicycle, a bus, or a sports car will result in differences in this force. Thus, if we want to use this as a measure of curvature, we must drive along the road with a fixed velocity. We add the restriction that $|\dot{r}| = 1$ to account for this.

1.3 Natural Parametrization

Definition. Natural Parameter

A parameter t on a curve $r(t)$ is called **natural** if $\left|\frac{dr}{dt}\right| = 1$.

Lemma. Let $r(t)$ be a curve. Fix $t_0 \in [a, b]$. Consider the signed length starting at $r(t_0)$. The curve has a natural orientation with respect to t_0. Recall that to compute arclength, we have

$$\ell(t) = \int_{t_0}^{t} |\dot{r}(\tau)| d\tau = \int_{t_0}^{t} \left|\frac{dr}{d\tau}\right| d\tau$$

Then ℓ is a natural parameter on this curve, that is

$$\left|\frac{dr}{d\ell}\right| = 1$$

Proof. Notice that

$$\frac{d\ell}{dt} = \left|\frac{dr}{dt}\right|$$

by the First Fundamental Theorem of Calculus. Further, $\frac{dr}{dt} \neq 0$ since the curve is regular. Since we have chosen signed length and $t \geq t_0$, we can conclude that $\frac{d\ell}{dt} > 0$. By the inverse function theorem, there exists $t = t(\ell)$ satisfying

$$\frac{dt}{d\ell} = \left(\frac{d\ell}{dt}\right)^{-1}$$

Substituting $t(\ell)$ into the parameter t, we have $r(t) = r(t(\ell))$, that is r as a function of ℓ. We note that distinct parametrizations are different as maps, but the image (i.e. the *trace*) is the same. Now, we compute that

$$\left|\frac{dr}{d\ell}\right| = \left|\frac{dr}{dt} \cdot \frac{dt}{d\ell}\right| = \left|\frac{dr}{dt}\right| \cdot \underbrace{\frac{dt}{d\ell}}_{>0} = \left|\frac{dr}{dt}\right| \cdot \left(\frac{d\ell}{dt}\right)^{-1} = \left|\frac{dr}{dt}\right| \cdot \left|\frac{dr}{dt}\right|^{-1} = 1$$

■

Definition. Reparametrization

If $t = t(s)$ is a function with $\frac{dt}{ds} \neq 0$, then $r(s) = r(t(s))$ is called a **reparametrization** of a curve $r(t)$.

Any curve may be reparametrized with a natural parameter, and we have chosen length as a natural parameter. Is it possible to describe other natural parameters?

Proposition. Let s be a natural parameter. Then $s = \pm\ell + \text{const}$.

Proof. Observe that

$$\ell(s) = \int_{s_0}^{s} \underbrace{\left|\frac{dr}{ds}\right|}_{=1} ds = \int_{s_0}^{s} ds = s - s_0$$

We consider positive and negative values of ℓ because it is possible that our bounds of integration go in a different direction (i.e. from s_0 to lower values). ■

The preceding proposition shows us that up to some constant, all natural parameters are length.

Definition. Curvature

The **curvature** of a curve is

$$k = \left|\frac{d^2 r}{ds^2}\right|$$

where s is a natural parameter.

Remark. Since curvature involves the second derivative, shifting our curve $r(t)$ by a constant vector does not change curvature, as taking the first derivative eliminates this constant.

Proposition. The curvature does not depend on the choice of a natural parameter.

Proof. Since $s = \pm\ell + c$, it follows that $\frac{d^2}{ds^2} = \frac{d^2}{d\ell^2}$.

To be precise, taking the derivative of s with respect to ℓ, we have $\frac{ds}{d\ell} = \pm 1$ and by the chain rule, $\frac{d}{d\ell} = \frac{d}{ds}\frac{ds}{d\ell}$. This shows us that $\frac{d}{d\ell} = \pm\frac{d}{ds}$.

■

Examples.

1. A line in $\mathbb{R}^n$. The usual parametrization is given by

$$r(t) = A + t\vec{v}$$

where A is some point in $\mathbb{R}^n$. Taking the derivative, we have $\dot{r}(t) = \vec{v}$ and taking the magnitude, we have $|\dot{r}(t)| = |\vec{v}|$. In general, the length of $\vec{v}$ is not 1. A simple way to modify this is to reduce our speed in proportion to $|\vec{v}|$. Thus, a natural parametrization is

$$r(s) = A + s \cdot \frac{\vec{v}}{|\vec{v}|}$$

since $|\frac{dr}{ds}| = \left|\frac{\vec{v}}{|\vec{v}|}\right| = 1$. Calculating the curvature, we have

$$k = \left|\frac{d^2 r}{ds^2}\right| = |\vec{0}| = 0$$

2. A circle in a plane. Assume that the center of our circle is at the origin. The usual

parametrization of a circle and its velocity are:

$$r(t) = (R\cos t, R\sin t)$$
$$\frac{dr}{dt} = (-R\sin t, R\cos t)$$
$$\left|\frac{dr}{dt}\right| = R$$

Notice that the speed is constant. So, to travel with velocity 1, we may simply travel R times slower. Consider the reparametrization

$$r(s) = \left(R\cos\left(\frac{s}{R}\right), R\sin\left(\frac{s}{R}\right)\right)$$
$$\frac{dr}{ds} = \left(-\sin\left(\frac{s}{R}\right), \cos\left(\frac{s}{R}\right)\right)$$

Thus, $\left|\frac{dr}{ds}\right| = 1$ and s is a natural parameter. Further

$$\frac{d^2r}{ds^2} = -\frac{1}{R}\left(-\sin\left(\frac{s}{R}\right), \cos\left(\frac{s}{R}\right)\right)$$

So, we compute curvature as

$$k = \left|\frac{d^2r}{ds^2}\right| = \frac{1}{R}$$

From a geometric point of view, this means that that larger the radius of the circle R, the less curvature it will have at any given point.

1.3.1 Limitations of Computing Curvature with Natural Parameter

Unfortunately, the difficulty in calculating curvature lies in determining the natural parameterization. In the previous examples, since velocity is constant, it is simple to invert this by division. In the general case, we have a length computed as

$$\ell(t) = \int_{t_0}^{t} \left|\frac{dr}{d\tau}\right| d\tau$$

and we want to find the inverse function $t = t(\ell)$. In many cases, evaluating an arbitrary integral may be nontrivial, and afterwards, we still need to determine the inverse function. From this perspective, it seems that the definition of curvature is very limited, however, it is possible to derive curvature for an arbitrary parameter. To summarize, this approach for determining curvature using a natural parameter is convenient for proofs, but it is impractical for actual computations.

1.4 Frenet Formulae in the Plane

To write these formulas, we must introduce the concept of a Frenet frame. Let $\mathbf{r}(s)$ be a planar curve and s a natural parameter. Call $\frac{d\mathbf{r}}{ds} = v(s)$. Then,

$$|v| = \left|\frac{d\mathbf{r}}{ds}\right| = 1$$

since s is a natural parameter. So, the velocity vector has constant length 1.
Lemma. If $|a(t)| = 1$, then $\dot{a} \perp a$. Since $|a(t)| = 1$, this means that $\langle a, a\rangle = 1$. Taking the derivative on both sides, we see that $\frac{d}{dt}\langle a, a\rangle = 0$. But by the bilinearity and symmetry of the scalar product, we have

$$0 = \frac{d}{dt}\langle a, a\rangle = \langle \dot{a}, a\rangle + \langle a, \dot{a}\rangle = 2\langle \dot{a}, a\rangle$$

Thus, $\langle \dot{a}, a\rangle = 0$ and $\dot{a} \perp a$.

■

Geometrically, since $|a(t)| = 1$, this means that the endpoint of the vector $a(t)$ coincides with some unit circle. Also, we know that the velocity vector $\dot{a}$ must be tangent to a and that the tangent vector to any point on a circle is perpendicular to the radius at that point. So again we have $a \perp \dot{a}$.
From the lemma, we conclude that $|v| = 1 \implies \dot{v} \perp v$. This means that

$$\frac{d^2\mathbf{r}}{ds^2} \perp \frac{d\mathbf{r}}{ds}$$

Let consider the unit vector

$$n(s) := \frac{d^2\mathbf{r}}{ds^2} \Big/ \left|\frac{d^2\mathbf{r}}{ds^2}\right| = \frac{1}{k} \cdot \frac{d^2\mathbf{r}}{ds^2}$$

From this, we obtain an orthonormal basis $v(s), n(s)$ for a planar curve.

- The vector $v(s)$ is called the **tangent vector** and it may also be denoted by t, T.
- The vector $n(s)$ is called the **normal vector** and it may also be denoted by N.
- We call the basis $v(s), n(s)$ a **Frenet basis**. When we consider the point $\mathbf{r}(s)$ and the triple $\mathbf{r}(s), v(s), n(s)$, this is called a **Frenet frame**.
- As we travel along our curve $\mathbf{r}(s)$, the Frenet frame travels as well, and for this reason such objects are often called *moving frames*.

It turns out that the Frenet frame gives us very important formulas, as one may consider many of the theorems of differential geometry to be generalizations of the Frenet formulas. The Frenet formulas involve taking derivatives of the vectors that make up the Frenet basis.

Theorem. Frenet Formulae in $\mathbb{R}^2$.

$$\frac{dv}{ds}(s) = k(s)n(s)$$
$$\frac{dn}{ds}(s) = -k(s)v(s)$$

Often, we may write the formulae as $\dot{v} = kn$, $\dot{n} = -kv$, omitting arguments for brevity.

Proof. Observe that

$$\frac{dv}{ds} = \frac{d}{ds}\left(\frac{dr}{ds}\right) = \frac{d^2r}{ds^2} = kn$$

Since $|n| = 1$, by the lemma we see that $\dot{n} \perp n$. Recall that $v \perp n$ and since we are dealing with planar curves, we must have

$$\dot{n} = \lambda v$$

as these two vectors are parallel. Additionally, since $n \perp v$, we have $\langle n, v\rangle = 0$ and taking the derivative on both sides, $\frac{d}{ds}\langle n, v\rangle = 0$. Further, by the product rule, we see that

$$\left\langle \frac{dn}{ds}, v \right\rangle + \left\langle n, \frac{dv}{ds} \right\rangle = 0$$
$$\langle \lambda v, v\rangle + \langle n, kn\rangle = 0$$
$$\lambda + k = 0$$

Thus, we have $\lambda = -k$. We obtain the above formulas by substitution.

■

Proposition. The curvature is the angular velocity of rotation of the tangent line, that is

$$k(s) = \lim_{\varepsilon\to 0} \frac{\alpha(s,\varepsilon)}{\varepsilon}$$

Proof. Recall that we may write the derivative as

$$f(s+\varepsilon) = f(s) + \varepsilon \dot{f}(s) + O(\varepsilon)$$

This is just the difference quotient rearranged, or a Taylor formula. Using the Frenet formulas, we may write

$$v(s+\varepsilon) = v(s) + \varepsilon k(s)n(s) + O(\varepsilon)$$
$$n(s+\varepsilon) = n(s) - \varepsilon k(s)v(s) + O(\varepsilon)$$

Also, recall that

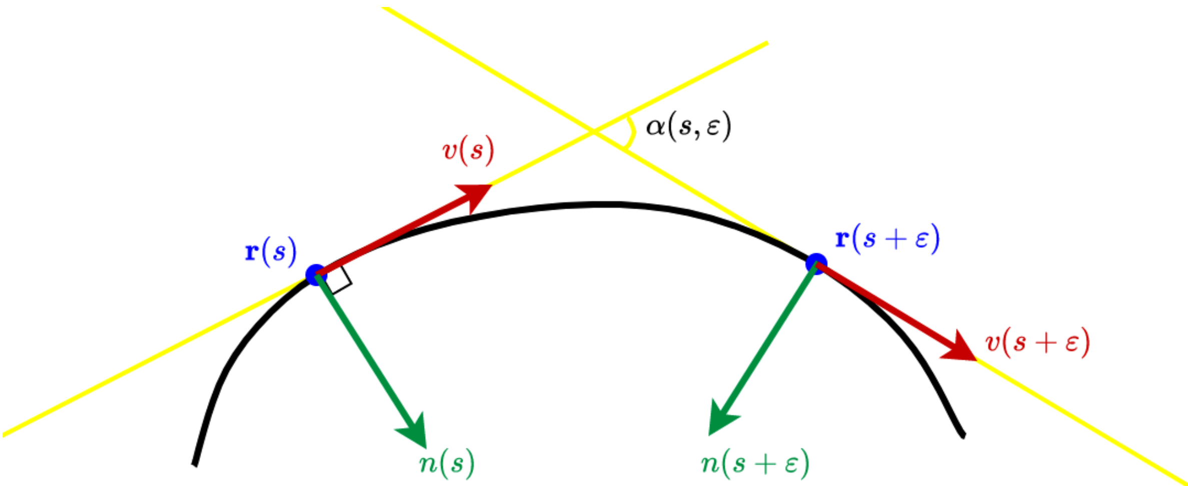
$$\cos\alpha = 1 + O(\alpha)$$
$$\sin\alpha = \alpha + O(\alpha)$$

These are just the first terms in the Taylor approximation. Substituting in α, we have

$$1 = \cos(\varepsilon k(s)) + O(\varepsilon)$$
$$\varepsilon k(s) = \sin(\varepsilon k(s)) + O(\varepsilon)$$

Then, we may write by substitution again,

$$v(s+\varepsilon) = \cos(\varepsilon k(s)) \cdot v(s) + \sin(\varepsilon k(s)) \cdot n(s) + O(\varepsilon)$$
$$n(s+\varepsilon) = -\sin(\varepsilon k(s)) \cdot v(s) + \cos(\varepsilon k(s)) \cdot n(s) + O(\varepsilon)$$



The above coefficients are suggestive: when we change between two orthonormal bases, the transformation must involve an orthogonal matrix. In our case, the transition from the orthonormal basis $v(s), n(s)$ to $v(s+\varepsilon), n(s+\varepsilon)$ is given by a rotation by some angle $\alpha(s, \varepsilon)$, that is by a matrix:

$$\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

This means that the angle is related to the curvature by $\alpha(s, \varepsilon) = \varepsilon \cdot k(s) + O(\varepsilon)$.

■

1.5 Frenet Formulae in Space

1.5.1 On the Vector Product

When we operate in $\mathbb{R}^3$, there is a choice of orientation to be made. When we equip $\mathbb{R}^3$ with an orientation, there exists an operation specific to $\mathbb{R}^3$ called the **cross product** or **vector product**.

- For vectors any $\vec{v}, \vec{w} \in \mathbb{R}^3$, the vector product is denoted by:

  $$\vec{v} \times \vec{w} \text{ or } [\vec{v}, \vec{w}]$$

  The vector $[\vec{v}, \vec{w}]$ is orthogonal to both $\vec{v}$ and $\vec{w}$.

- Further,
$$|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\theta)$$
this is also the area of the parallelogram formed by $\vec{v}$ and $\vec{w}$, assuming $0 \leq \theta \leq \pi$.
- $\vec{v}, \vec{w}, \vec{v} \times \vec{w}$ is a positive oriented system (i.e. right-hand system).
- Consider vectors v, n and $b = [v, n]$. Since v and n are orthonormal, the parallelogram they form is a square with area 1, so $|b| = 1$. Further, $b \perp v$ and $b \perp n$. Hence, v, n, b is a **positive oriented orthonormal basis**.
- The vector $b = [v, n]$ is called a **binormal vector**, also denoted by B.
- v, n, b form a **Frenet basis** and $r(s), v(s), n(s), b(s)$ form a **Frenet frame**.

In the next section, we will discover that curvature is not enough to describe curves in space. For planar curves, we have only curvature, but for space curves, we have curvature and torsion.

1.5.2 Deriving Frenet Formulae in $\mathbb{R}^3$

From our previous discussion, we know

$$\begin{aligned} \dot{v} &= \frac{d}{ds}\frac{dr}{ds} = \frac{d^2r}{ds^2} = kn \\ \dot{n} &= \lambda v + \varkappa b \end{aligned}$$

Since $|n| = 1$ by construction, by the lemma we know that $\dot{n} \perp n$. Since $\dot{n} \perp n$, and v, n, b is a Frenet basis, we must have $\dot{n} \in \text{span}\{v, b\}$ and this justifies the second equation. Now, we see that $(v, n) = 0 \implies \frac{d}{ds}(v, n) = 0$. Also

$$\begin{aligned} (\dot{v}, n) + (v, \dot{n}) &= 0 \\ (kn, n) + (v, \lambda v + \varkappa b) &= 0 \\ k + \lambda &= 0 \end{aligned}$$

As before, we have $\lambda = -k$ and we obtain the expression $\dot{n} = \lambda v + \varkappa b$. The next question involves how to find $\varkappa$. As it turns out, $\varkappa$ cannot be expressed in terms of curvature - it is an independent quantity.

Definition. Torsion

A **torsion** $\varkappa$ is given by the formula $\varkappa = (\dot{n}, b)$ where $\dot{n}$ indicates the derivative with respect to a natural parameter s. We make this definition since

$$(\dot{n}, b) = (\lambda v + \varkappa b, b) = \varkappa(b, b) = \varkappa$$

Then, from the bilinearity of the vector product, we have

$$\begin{aligned}\dot{b} &= \frac{d}{ds}[v, n] = [\dot{v}, n] + [v, \dot{n}] \\ &= [kn, n] + [v, -kv + \varkappa b] \\ &= 0 + \varkappa[v, b]\end{aligned}$$

Since v, n, b is positively oriented, then v, b, n is negatively oriented, thus $[v, b] = -n$.

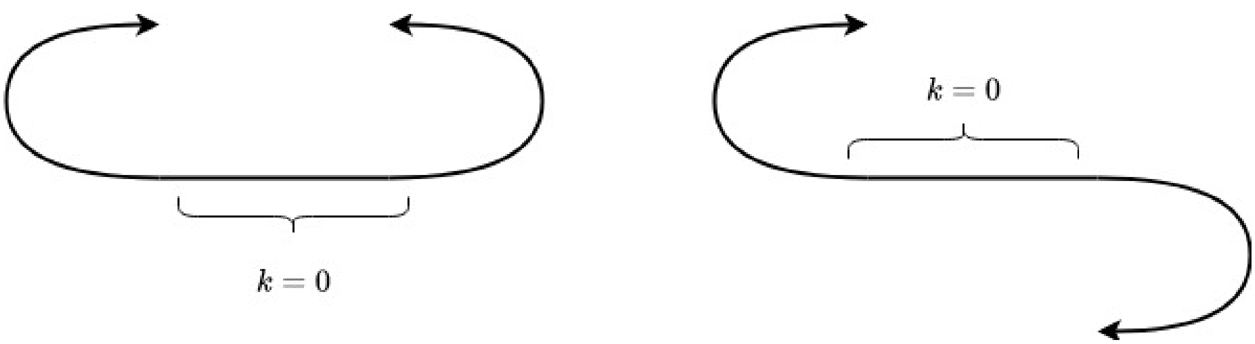
Theorem. Frenet Formulae in $\mathbb{R}^3$.

$$\begin{aligned}\frac{dv}{ds}(s) &= k(s)n(s) \\ \frac{dn}{ds}(s) &= -k(s)v(s) + \varkappa(s)b(s) \\ \frac{db}{ds}(s) &= -\varkappa(s)n(s)\end{aligned}$$

One important idea is that curvature and torsion define our curve uniquely up to isometry. Intuitively, isometries do not affect curvature because constant shifts are eliminated when we take the second derivative, and orthogonal transformations do not change length. This idea also applies to torsion.

1.6 Curvature and Torsion Define a Spatial Curve Uniquely up to Isometry

Given arbitrary functions f, g is there a curve satisfying $k(s) = f(s)$ and $\varkappa(s) = g(s)$?
The answer is not necessarily. We need to observe a condition. First, $k = |\frac{d^2r}{ds^2}| \geq 0$. But, we need the stronger condition $k > 0$. When curvature vanishes, we cannot define the Frenet frame. Another issue is that curves are no longer unique if we do not require positive curvature.



Here, both curves above have the same curvature as the right hand section of both curves is just as reflection of the other and again, orthogonal transformations preserve length of vectors (i.e. since it preserves dot products and since $\dot{v}$ is a vector). However, these curves are clearly distinct. Thus, this example shows us that we should require $f > 0$.

Theorem. Let $f, g : [a, b] \to \mathbb{R}$ be two smooth functions and $f > 0$. Then, there exists a curve $r : [a, b] \to \mathbb{R}^3$ such that $k(s) = f(s)$, $\varkappa(s) = g(s)$. Moreover, this curve is unique up to an isometry of the Euclidean space. There is an analogous, simpler result for planar curves.

Notice that the statement of this theorem implies the independence of torsion from curvature, as both are arbitrarily chosen functions.

Proof.

Our goal will be to construct a curve r satisfying the conditions of the theorem. Recall the Frenet formulae.

$$\begin{cases} \dot{v} = kn \\ \dot{n} = -kv + \varkappa b \\ \dot{b} = -\varkappa n \end{cases}$$

We also have by definition that $\dot{r} = v$. Consider the following system of first-order ODEs:

$$\begin{cases} \dot{v} = fn \\ \dot{n} = -fv + gb \\ \dot{b} = -gn \\ \dot{r} = v \end{cases}$$

Note that f, g are given by the hypothesis. Meanwhile, v, n, b, r are unknown vector functions, that is, the components of each function, $(v_1, v_2, v_3, n_1, n_2, n_3, \ldots)$ are unknown. Let us choose $s_0 \in [a, b]$, a point $r_0 \in \mathbb{R}^3$, and a positively oriented orthonormal basis: v_0, n_0, b_0. To make this into a Cauchy problem, we add the initial conditions

$$\begin{cases} v(s_0) = v_0 \\ n(s_0) = n_0 \\ b(s_0) = b_0 \\ r(s_0) = r_0 \end{cases}$$

An important point is that this is a Cauchy problem for system of **linear** ODEs. Generally, there exists a unique solution for the Cauchy problem around some neighborhood around the initial data. However, when it is a system of linear ODEs, this unique solution exists over the entire interval (global existence and uniqueness). Hence, there exists a unique solution $r(s), v(s), n(s), b(s)$.

Next, we aim to show that $r(s), v(s), n(s), b(s)$ is a positively oriented orthonormal frame. To each set of vectors, we can define the Gram matrix G, which is the sqaure matrix consisting of pairwise scalar products of the vectors, i.e. $G_{ij} = \langle \vec{v_i}, \vec{v_j} \rangle$. Note that G must be symmetric due to the commutativity of the scalar product. Consider the Gram matrix formed from the vectors $v(s), n(s), b(s)$. Thus, we have $G_{11} = (v, v), G_{12} = (v, n), G_{13} = (v, b)$ and so forth. Taking the derivatives, we see that

$$\begin{aligned}
\dot{G}_{11} &= \frac{d}{ds}(v,v) = (\dot{v}, v) + (v, \dot{v}) = 2f(n, v) = 2fG_{12} \\
\dot{G}_{12} &= \frac{d}{ds}(v,n) = (\dot{v}, n) + (v, \dot{n}) = (fn, n) + (v, -fv + gb) \\
&= fG_{22} - fG_{11} + gG_{13} \\
&\dots
\end{aligned}$$

Hence G_{ij} satisfy a system of linear ODEs. Also, we see that

$$\begin{aligned}
G_{11}(s_0) &= (v(s_0), v(s_0)) = (v_0, v_0) = 1 \\
G_{12}(s_0) &= (v(s_0), n(s_0)) = (v_0, n_0) = 0 \\
&\dots
\end{aligned}$$

With the Gram matrix and the matrix of its derivatives, we obtain a Cauchy problem for a system of linear ODEs. Given the above initial conditions, observe that, for distinct $i, j \in \mathbb{N}$,

$$G_{ii}(s) \equiv 1 \text{ and } G_{ij} \equiv 0$$

is a solution. Since this solution is unique, this implies that for any s, $v(s), n(s), b(s)$ form an orthonormal basis. Since this triple is an orthonormal basis, we see that $|\frac{dr}{ds}| = |v(s)| = 1$, which implies that s is a natural parameter on $r(s)$. Now, since s is a natural parameter and f is strictly positive,

$$k = \left| \frac{d^2 r}{ds^2} \right| = |\dot{v}| = |fn| = f|n| = f$$

Next, for torsion, we see

$$\begin{aligned}
\varkappa &= (\dot{n}, b) = (-fv + gb, b) \\
&= -f(v, b) + g(b, b) \\
&= -fG_{13} + gG_{33} = g
\end{aligned}$$

Finally, the curve $r(s)$ is unique up to isometry due to the choice of s_0, r_0, and orthonormal basis we made in the beginning. If we fix our orthonormal basis and s_0 and consider different choices of r_0, then this simply translates the curve obtained as a solution. Fixing everything except our orthonormal basis results in curves that are related by rotations. Fixing everything except s_0 results in some isometry.

■

1.7 Generalization to Curves in $\mathbb{R}^n$

We generalize the Frenet formulas to $\mathbb{R}^n$. First, choose an orientation of $\mathbb{R}^n$.

> *Aside.* In $\mathbb{R}^n$, the standard orientation is defined by the standard basis $\{e_1, \ldots, e_n\}$ and it is considered **positively oriented**. For an arbitrary basis $\{v_1, \ldots, v_n\}$, the determinant of the matrix where each column is one of the vectors v_i defines orientation. If the determinant is positive, then this basis is positively oriented, and it is negatively oriented otherwise.

Let s be a natural parameter. We denote

$$\varepsilon_1 = \frac{dr}{ds}, \; \varepsilon_2 = \frac{d^2r}{ds^2}, \; \ldots, \varepsilon_n = \frac{d^nr}{ds^n}$$

From here, we want to obtain a system of vectors similar to the Frenet frame we have seen in $\mathbb{R}^2$ and $\mathbb{R}^3$. We assume that $\varepsilon_1, \ldots, \varepsilon_n$ are linearly independent. However, we do not have a vector product for all $\mathbb{R}^n$ which allows us to produce a basis of orthogonal vectors. The idea is to use **Gram-Schmidt orthogonalization** to produce orthogonal vectors. We have $\widetilde{\varepsilon_1} = \varepsilon_1$ and for subsequent vectors

$$\widetilde{\varepsilon_k} = \varepsilon_k - \sum_{i=1}^{k-1} \frac{(\varepsilon_k, \widetilde{\varepsilon_i})}{(\widetilde{\varepsilon_i}, \widetilde{\varepsilon_i})} \widetilde{\varepsilon_i}$$

Next, we normalize the vectors:

$$f_i = \frac{\widetilde{\varepsilon_i}}{|\widetilde{\varepsilon_i}|}$$

So, if $\varepsilon_1, \ldots, \varepsilon_n$ form a basis, then $f_1, \ldots, f_n$ form an orthonormal basis.

Since $f_1(s), \ldots, f_n(s)$ and $f_1(s+\varepsilon), \ldots, f_n(s+\varepsilon)$ are two orthonormal bases, the matrix of transformation between these two bases must be orthogonal.

> *Aside.* Let $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ be two orthonormal bases on an inner product space and let T be the linear transformation defined by $T(v_i) = w_i$. Further, for $i, j \in \{1, 2, \ldots, n\}$ we have $T(v_i) \cdot T(v_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta, showing that $[T]^t[T] = I_n = [T][T]^t$. Thus, the matrix of transformation between orthonormal bases is orthogonal.

Let us call this orthogonal matrix $A(s, \varepsilon)$. By the definition of orthogonal matrix, we have

$$A(s, \varepsilon)A^T(s, \varepsilon) = I$$

Further $\lim\limits_{\varepsilon \to 0} A(s, \varepsilon) = I$, since $\lim\limits_{\varepsilon \to 0} f_i(s+\varepsilon) = f_i(s)$. An orthogonal matrix may only take on values from $\{-1, 1\}$. We know that $\det(A(s, 0)) = \det(I) = 1$. From the continuity of the determinant and of A, we see that $\det(A(s,\varepsilon)) = 1$ for all $\varepsilon > 0$. This shows us why A must be a rotation in $\mathbb{R}^2$.

Remark. Taking the derivative, we see that

$$\frac{\partial}{\partial\varepsilon}(A(s,\varepsilon)A^T(s,\varepsilon)) = 0$$
$$\frac{\partial A}{\partial\varepsilon}(s,\varepsilon)A^T(s,\varepsilon) + A(s,\varepsilon)\frac{\partial A^T}{\partial\varepsilon}(s,\varepsilon) = 0$$

Plugging in $\varepsilon = 0$, we have

$$\frac{\partial A}{\partial\varepsilon}(s,0) + \frac{\partial A^T}{\partial\varepsilon}(s,0) = 0$$

Hence $B(s) := \frac{\partial A}{\partial\varepsilon}(s,0)$ is a skew-symmetric matrix, that is $B^T(s) = -B(s)$.

Recall that the columns of the change of basis matrix A^T are vectors of the original basis $\{f_i(s+\varepsilon)\}$ written in terms of the new basis $\{f_i(s)\}$. As a result, we can see that

$$\begin{aligned} f_1(s+\varepsilon) &= A^T_{11}f_1(s) + A^T_{21}f_2(s) + \ldots + A^T_{n1}f_n(s) \\ &= A_{11}f_1(s) + A_{12}f_2(s) + \ldots + A_{1n}f_n(s) \end{aligned}$$

Applying $\frac{\partial}{\partial\varepsilon}$ and taking $\varepsilon = 0$, we see that

$$\begin{aligned} \dot{f}_1(s) &= B_{11}(s)f_1(s) + B_{12}(s)f_2(s) + \ldots + B_{1n}(s)f_n(s) \\ \dot{f}_2(s) &= B_{21}(s)f_1(s) + B_{22}(s)f_2(s) + \ldots \end{aligned}$$

We claim that the expressions for $\dot{f}_k(s)$ are exactly the Frenet formulae in $\mathbb{R}^n$. To show this, first observe that

$$\varepsilon_k = \frac{d^k r}{ds^k}, \quad \widetilde{\varepsilon}_k = \frac{d^k r}{ds^k} + \alpha_{k1}\frac{dr}{ds} + \ldots + \alpha_{kk-1}\frac{d^{k-1}r}{ds^{k-1}}$$

What this means is that $\widetilde{\varepsilon}_k$ depends on derivatives of order up to k. The same holds for f_k, which just differs by a normalization constant:

$$f_k = \beta_{k1}\frac{dr}{ds} + \beta_{k2}\frac{d^2r}{ds^2} + \ldots + \beta_{kk}\frac{d^k r}{ds^k}$$

By the product rule, notice that

$$\dot{f}_k = \gamma_{k1}\frac{dr}{ds} + \ldots + \gamma_{kk+1}\frac{d^{k+1}r}{ds^{k+1}}$$

Remark. Above, we have seen how we can express f_k as a function of derivatives of r up to order k. We may reverse this relationship to express each derivative as a function of f_k. To be clear, we may express the relationship as

$$\begin{pmatrix} \beta_{11} & \beta_{12} & \ldots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \ldots & \beta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{n1} & \beta_{n2} & \ldots & \beta_{nn} \end{pmatrix} \begin{pmatrix} r^{(1)} \\ r^{(2)} \\ \vdots \\ r^{(n)} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

Further, since each f_k only depends on derivatives of r up to order k, we see that our transformation matrix is actually triangular:

$$\begin{pmatrix} \beta_{11} & 0 & \dots & 0 \\ \beta_{21} & \beta_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{n1} & \beta_{n2} & \dots & \beta_{nn} \end{pmatrix}$$

Since f_i are functions of derivatives up to order i, we may rewrite the previous equation for $\dot{f}_k$ as

$$\dot{f}_k = \tau_{k1} f_1 + \dots \tau_{kk+1} f_{k+1}$$

Letting $*$ indicate a nonzero entry, we see that

$$B(s) = \begin{pmatrix} * & * & 0 & \dots & 0 \\ * & * & * & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ * & * & \dots & * & * \\ * & * & \dots & * & * \end{pmatrix}$$

meaning that $B(s)$ is a lower triangular matrix with one diagonal above the diagonal. But recall the $B^T = -B$, so it follows that

$$B = \begin{pmatrix} 0 & k & & & \\ -k & 0 & \varkappa_1 & & \\ & -\varkappa_1 & 0 & \ddots & \\ & & \ddots & \ddots & \varkappa_{n-2} \\ & & & -\varkappa_{n-2} & 0 \end{pmatrix}$$

From this, we realize the generalization of the Frenet formulae. Here, k is called curvature and $\varkappa_1, \dots \varkappa_{n-2}$ are called higher torsions. As an example, notice that in $\mathbb{R}^n$, the Frenet formulae are given by

$$\begin{pmatrix} 0 & k & 0 \\ -k & 0 & \varkappa \\ 0 & -\varkappa & 0 \end{pmatrix} \begin{pmatrix} v \\ n \\ b \end{pmatrix} = \begin{pmatrix} \dot{v} \\ \dot{n} \\ \dot{b} \end{pmatrix}$$

■

Aside. Orthogonal transformations preserve dot products and lengths. Notice that for any two vectors u, v, we have

$$(Au, Av) = (Au)^T(Av) = u^T A^T A v = u^T v = (u, v)$$

2 Surfaces

A circle has constant curvature, and one may intuit this from its rotational symmetry. By comparison, one might expect an ellipse to have different curvature at different points (i.e. along where the semi-axes intersect the shape). While isometries preserve curvature, an arbitrary affine transformation may not preserve curvature. This illustrates the connection between curvature and Euclidean structure.

There are two ways to describe surfaces. There is an **implicit description**, $F(x, y, z) = 0$. Also, there is a **parametric description**, given below.

Definition. Surface

A regular, smooth parametric surface is a map $r : \mathscr{D} \to \mathbb{R}^3$, with $\mathscr{D} \subseteq \mathbb{R}^2$ being open, such that:

1. $r \in \mathscr{C}^\infty(\mathbb{R}^2, \mathbb{R}^3)$
2. We require at each point $(u, v) \in \mathscr{D}$ that

$$\operatorname{rank}\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} = 2$$

Note that our map r is explicitly parametrized as $r(u, v) = (x(u, v), y(u, v), z(u, v))$. Also

$$r_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \text{ and } r_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$$

The vector of partial derivatives are the columns of the matrix involved in the regularity condition. The requirement that the rank of this matrix is 2 is equivalent to saying that r_u and r_v are linearly independent. This condition of linear independence is analogous to our requirement that regular curves should have non-vanishing derivatives at all points of their domains.

Definition. Reparametrization

If $u = u(s, t)$ and $v = v(s, t)$ is a smooth bijection and

$$J = \det\begin{pmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{pmatrix} \neq 0$$

then $r(s, t) = r(u(s, t), v(s, t))$ is called a **reparametrization** of $r(u, v)$. Here, $r(u, v) = (x(u, v), y(u, v), z(u, v))$ and the same holds for $r(s, t)$.

Proposition. If $r(u,v)$ is regular and smooth, then $r(s,t)$ is also a regular and smooth surface.

Proof. By the chain rule

$$\frac{\partial r}{\partial s} = \frac{\partial r}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial r}{\partial v}\frac{\partial v}{\partial s}$$
$$\frac{\partial r}{\partial t} = \frac{\partial r}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial r}{\partial v}\frac{\partial v}{\partial t}$$

are smooth (product and sum of smooth functions). Further, notice that

$$\begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{pmatrix}$$

Since $r(u,v)$ is regular, we see that the first matrix on the right-hand side has rank 2. The second matrix on the right-hand side, J, is invertible, so it must have full rank. Further, recall that if B is invertible, then $\operatorname{rank}(AB) = \operatorname{rank}(A)$. Thus, we see that the matrix on the left has rank 2. So $r(s,t)$ is regular and smooth.

2.1 Different Perspectives on Maps of Geometers and Analysts or Algebraists

Let $\mathbb{E}^2$ be the Euclidean plane. Pick an arbitrary point $O \in \mathbb{E}^2$. Define a function $f : \mathbb{E}^2 \to \mathbb{R}$ by

$$f(P) = \left|\overrightarrow{OP}\right|^2$$

At this point, there are only geometric objects. Suppose we add in Cartesian coordinates, so that the point P is located at (x,y). Now, call

$$f(x,y) = \sqrt{x^2+y^2}$$

this will be the length of the vector OP. Also, in polar coordinates, we have $f(r,\varphi) = r$. For geometers, there is a geometric interpretation or meaning associated with every formula. For an algebraist, these are just symbolic objects that may be manipulated freely. For instance, we could plug in r and φ into the first formula to get $f(r,\varphi) = \sqrt{r^2+\varphi^2}$. In algebra or analysis, a function is treated as a computational description. Is it possible to unify the geometric and algebraic perspectives?

Consider the following commutative diagram

$$\begin{array}{ccccc} \mathbb{R}^2_{x,y} & \xrightarrow{\Phi} & \mathbb{E}^2 & \xrightarrow{f} & \mathbb{R} \\ & \overset{\Psi}{\nearrow} & & & \\ (\mathbb{R}_+)^2_{r,\varphi} & & & & \end{array}$$

From the above, we have $(f \circ \Phi)(x,y) = \sqrt{x^2+y^2}$ and $(f \circ \Psi)(r,\varphi) = r$ in the algebraic perspective. Because of this overly structural view, geometers prefer to make simplifications.

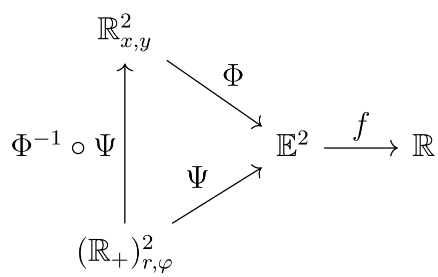
A geometer may state that f in coordinates (x, y) has the form $f(x, y) = \sqrt{x^2 + y^2}$, but they mean $f \circ \Phi$. An analogous statement is true for polar coordinates and $f \circ \Psi$.
Now, suppose that we have the change of coordinates from polar to rectangular:

$$x = x(r, \varphi) = r \cos \varphi$$
$$y = y(r, \varphi) = r \sin \varphi$$

Then, we have

$$f(r, \varphi) = f(x(r, \varphi), y(r, \varphi))$$

It is important to see that the map f on the right-hand side and left-hand side are different. Consider again the commutative diagram with this transformation



Making the reparametrization explicit, we have

$$(f \circ \Psi)(r, \varphi) = (f \circ \Phi)(\Phi^{-1} \circ \Psi)(r, \varphi) = (f \circ \Phi)(x, y)$$

So we see that $f \circ \Psi$ is distinct from $f \circ \Phi$. The key point is that in algebra or analysis, a function is defined symbolically by a letter. However, in geometry, the map is defined by both by the letter associated with the function and its arguments. Without arguments, then we are speaking about a geometric map without any coordinate system.

2.2 Terminology

Consider the diagram below:

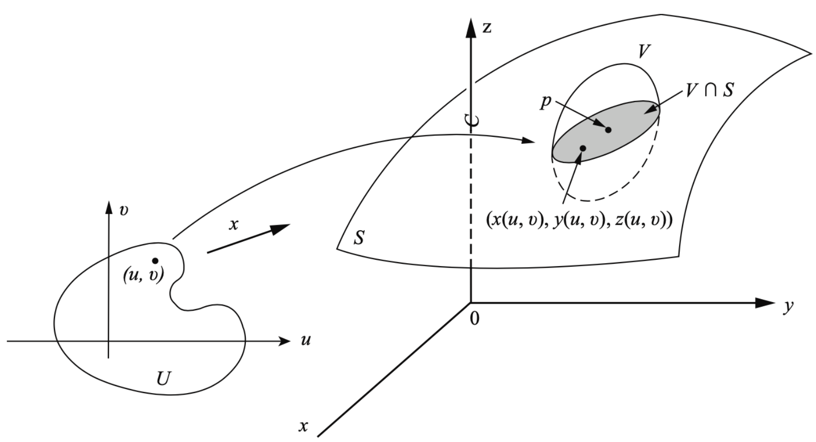


Figure 2-1

Definition. Coordinates

The coordinates u, v in the domain are called the **local coordinates**. The coordinates x, y, z are called **coordinates in the ambient space** or **global coordinates**. Letting $p(x, y, z)$ be the ambient coordinates, if $p = r(u, v)$, then u, v are considered local coordinates of p. Also, x, y, z are not arbitrary.

Aside. One more abuse of terminology is that we may call the image $r(\mathscr{D})$ a surface, even though we call its parametrization a surface too.

Proposition. Any point of a surface has a neighborhood such that, in this neighborhood, we can take x, y or x, z, or y, z as local coordinates.

Proof. Since the surface is regular, we know

$$\operatorname{rank}\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} = 2$$

Take a point on the surface. Because of the above fact and without loss of generality, there must exist a nondegenerate minor

$$\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \neq 0$$

at this point. The inverse function theorem implies that in some neighborhood, $u = u(x, y)$ and $v(x, y)$. So, there is a reparametrization $r(u(x, y), v(x, y)) = r(x, y)$ and

$$r(x, y) = (x, y, z(x, y))$$

This shows that a surface is locally a graph of a smooth function. We may think of this as converting from the parametric form to the explicit form around some neighborhood.

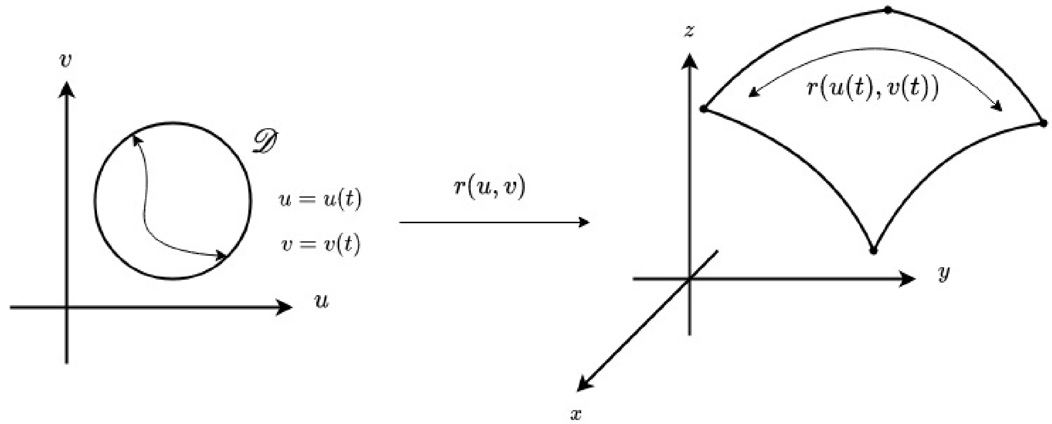
■

2.3 Curves Lying on a Surface

Recall that we proved that a curve is a straight line if and only if it has curvature 0. However, notice that there are no straight lines on a sphere. Thus, something in the geometry of a sphere eliminates curves with curvature 0. From this example, we see that the geometry of a surface imposes some conditions on the curvature of curves which lie on that surface. From this idea, Euler reasoned that to study curvature of a surface, we should study curvature of curves lying on the surface.

2.3.1 Two Descriptions of Curves on a Surface

There are two equivalent descriptions of curves that lie on surfaces, a local description and a global description.



Above, $u = u(t)$ and $v = v(t)$ parameterize a curve in **local coordinates**, which is then mapped to a curve in the ambient space, which lies on the surface

$$r(u, v) = (x, y, z)$$

where

$$x = x(u(t), v(t))$$
$$y = y(u(t), v(t))$$
$$z = z(u(t), v(t))$$

For the description of a curve in **global coordinates**, consider the parametrization

$$r(t) = (x(t), y(t), z(t))$$

Recall from the proposition, we may locally express $u = u(x, y)$ and $v = v(x, y)$. Plugging in our global parametrization, we obtain

$$u = u(x(t), y(t)) \text{ and } v = v(x(t), y(t))$$

This is a curve in local coordinates. The equivalence of these two descriptions means that one may characterize a curve in the ambient space lying on the surface given a curve in local coordinates, and contrarily, given a curve in the ambient space, we may return to a curve in local coordinates.

Exercise. Smoothness and regularity in local coordinates imply smoothness and regularity in global coordinates and vice versa.

Although these descriptions are equivalent, it is often convenient to work with local coordinates rather than global ones, because defining a curve on global coordinates means that we are subject to the restrictions imposed by the surface (i.e. consider a curve that must lie on a sphere). When defining a curve in local coordinates, we are only restricted by the domain.

2.3.2 Velocity Vector

Suppose we have $r(t) = r(u(t), v(t))$. How can we understand the velocity vector $\frac{dr}{dt}$ at a point? Note that this will be a velocity vector in the ambient space and **not** in the space of local coordinates. The chain rule implies

$$\dot{r} = r_u \dot{u} + r_v \dot{v}$$

We have emphasized the vector and scalar quantities. Note that $r_u = (\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u})$. To be explicit, the abbreviation above may be expanded as

$$\frac{dr}{dt}(t) = \frac{\partial r}{\partial u}(u(t), v(t))\frac{du}{dt} + \frac{\partial r}{\partial v}(u(t), v(t))\frac{dv}{dt}$$

There is a geometric viewpoint to the above formula.

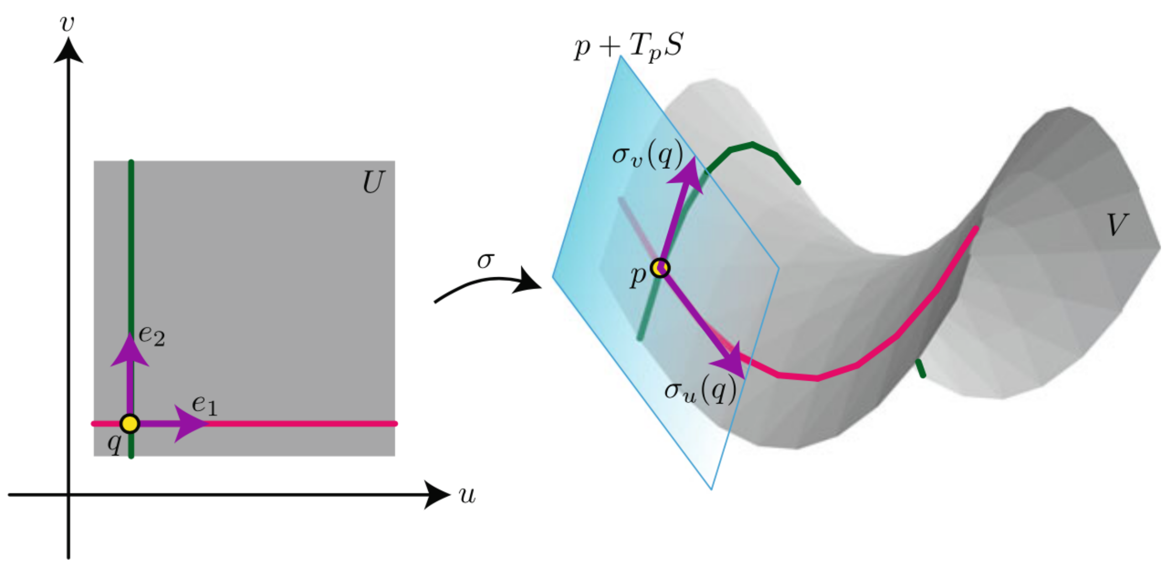


FIGURE 3.19. The translation to p of the subspace T_pS

Let $A = r(u_0, v_0)$ be a point on the surface, which means that A has local coordinates u_0, v_0 and let $u = u(t)$ and $v = v(t)$ parameterize a curve going through the point A. Let $u(t_0) = u_0$ and $v(t_0) = v_0$, then plugging this point into our equation derived from the chain rule, we have

$$\dot{r}(t_0) = r_u(A)\dot{u}(t_0) + r_v(A)\dot{v}(t_0)$$

Observe that $r_u(A)$ and $r_v(A)$ are vectors at the point A which do not depend on the curve going through A. Further, the regularity condition that

$$\text{rank}\begin{pmatrix} | & | \\ r_u(A) & r_v(A) \\ | & | \end{pmatrix} = 2$$

is equivalent to the fact that $r_u(A)$ and $r_v(A)$ are linearly independent, that is, they span a 2-dimensional plane.

Definition. Tangent Plane

The **tangent plane** of a surface Σ at the point A is given by

$$T_A\Sigma = \text{span}(r_u(A), r_v(A))$$

Thus, elements of the tangent plane are called **tangent vectors**.

Proposition. $T_A\Sigma$ consists exactly of velocity vectors of curves lying on Σ and passing through the point A. In other words, a curve lying on Σ passes through A if and only if its tangent vector lies in $T_A\Sigma$.

Proof. Suppose a curve lies on Σ and passes through $A = r(u_0, v_0)$. From the chain rule

$$\dot{r} = r_u\dot{u} + r_v\dot{v}$$

and $\dot{r} \in T_A\Sigma$. On the other hand, suppose that $V \in T_A\Sigma$. Then by the definition of tangent plane, we may write

$$V = \alpha r_u(A) + \beta r_v(A)$$

Define a curve in local coordinates by

$$u = u_0 + \alpha t$$
$$v = v_0 + \beta t$$

Taking the derivative with respect to t, we have

$$\dot{r} = r_u\dot{u} + r_v\dot{v} = r_u\alpha + r_v\beta = V$$

Therefore, r is a curve lying on Σ and passing through A which has tangent vector V.

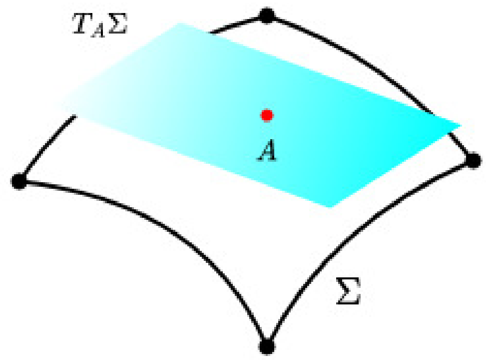
■

2.4 First Fundamental Form

Let $\langle\cdot,\cdot\rangle$ be the scalar product in Euclidean space, that is, for vectors $V, W \in \mathbb{R}^3$

$$\langle V, W\rangle = \sum_{i=1}^{3} V_i W_i$$

Recall that $\langle\cdot,\cdot\rangle$ is a quadratic form. Let Σ be a surface and A be some point on the surface. We also have the tangent plane $T_A\Sigma \subseteq \mathbb{R}^3$.



Above is a visualization of the tangent plane to the point on a surface.

Definition. First Fundamental Form

Let us restrict our scalar product to this tangent plane

$$\mathrm{I}_A = \langle \cdot, \cdot \rangle \Big|_{T_A\Sigma}$$

As a result, we obtain a quadratic form on the tangent space. This is called the **first quadratic (fundamental) form** of Σ at a point A. Note that the first fundamental form depends on the point A.

2.4.1 How do we write the matrix associated with $\mathbf{I}_A$ in a basis?

Recall that linear operators and quadratic forms constitute two major topics in the study of linear algebra. In fact, quadratic forms are symmetric bilinear forms.

1. $r_u(A), r_v(A)$ is a basis for the tangent plane.
   Exercise. If $u = u(s,t)$ and $v = v(s,t)$ is a change of local coordinates, then r_s, r_t is just another basis of the same tangent plane $T_A\Sigma$.
   Remark. r_u and r_v are analytical objects in that they depend on the parametrization. The choice of basis depends on the choice of local coordinates, but the tangent plane is independent of local coordinates.

2. Let $V, W \in \mathbb{R}^k$ and let $Q(V,W)$ be a quadratic form. We also denote $V = \sum_{i=1}^k V_i e_i$ and $W = \sum_{i=1}^k W_i e_i$. Then

$$Q = \begin{pmatrix} Q(e_1,e_1) & Q(e_1,e_2) & \cdots & Q(e_1,e_k) \\ \vdots & \cdots & \ddots & \vdots \\ Q(e_k,e_1) & Q(e_k,e_2) & \cdots & Q(e_k,e_k) \end{pmatrix}$$

How can we deduce that Q satisfies the above? Since Q is bilinear, we know that

$$\begin{aligned} Q(V,W) &= Q\left(\sum_{i=1}^{k} V_i e_i, \sum_{i=1}^{k} W_i e_i\right) \\ &= \sum_{i=1}^{k}\sum_{j=1}^{k} V_i W_j \cdot Q(e_i, e_j) \\ &= \begin{pmatrix} V_1 & \cdots & V_k \end{pmatrix} Q \begin{pmatrix} W_1 \\ \vdots \\ W_k \end{pmatrix} \end{aligned}$$

Hence, the matrix of I_A in the basis $r_u(A), r_v(A)$ is given by

$$\mathrm{I}_A = \begin{pmatrix} \langle r_u(A), r_u(A)\rangle & \langle r_u(A), r_v(A)\rangle \\ \langle r_u(A), r_v(A)\rangle & \langle r_v(A), r_v(A)\rangle \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

In this case, our Q is the scalar product as we have defined the first fundamental form this way. Keep in mind that $r_u = (\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u})$

What is the purpose of the first fundamental form? Recall that scalar products are used to find lengths of vectors and angles between vectors. In the same way, the first fundamental form serves to find lengths and angles.

Example. Recall that arclength of a curve is given by

$$\ell = \int_{t_0}^{t} \left|\frac{dr}{dt}\right| dt$$

How do we compute this in global and local coordinates?

Global Coordinates. We consider the function $r(t) = (x(t), y(t), z(t))$. Taking the derivative, we have $\dot{r}(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$. Then, we simply have

$$\ell(t) = \int_{t_0}^{t} \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2}\, dt$$

Local Coordinates. We have $r(t) = r(u(t), v(t))$. Hence

$$\dot{r} = r_u \dot{u} + r_v \dot{v}$$

So, $\dot{r}$ has coordinates $(\dot{u}, \dot{v})$ in the basis r_u, r_v. Further

$$\langle \dot{r}, \dot{r}\rangle = \mathrm{I}(\dot{r}, \dot{r}) = \begin{pmatrix} \dot{u} & \dot{v} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}$$

Since $\left|\frac{dr}{dt}\right| = \sqrt{\langle \dot{r}, \dot{r}\rangle}$, we conclude that

$$\ell(t) = \int_{t_0}^{t} \sqrt{\begin{pmatrix} \dot{u} & \dot{v} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}}\, dt$$

We emphasize again that entries in the first fundamental form like $E(u(t), v(t))$ depend on the local coordinates, but we omit arguments. This formula is simpler and more convenient if our curve is already given in local coordinates. To determine the angles between vectors, we may notice:

$$\cos\alpha = \frac{\langle V, W\rangle}{|V||W|} = \frac{\mathrm{I}_A(V, W)}{\sqrt{\mathrm{I}_A(V, V)\cdot \mathrm{I}_A(W, W)}}$$

Finally, the area of a surface may be computed as

$$\mathrm{Area}(\Sigma) = \iint_{\mathscr{D}} \sqrt{EG - F^2}\, du\, dv$$

The area of a rectangular section, considering the tangent vectors with respect to each of the local coordinates, is

$$\sqrt{\det\begin{pmatrix} E & F \\ F & G\end{pmatrix}} = \sqrt{EG - F^2}$$

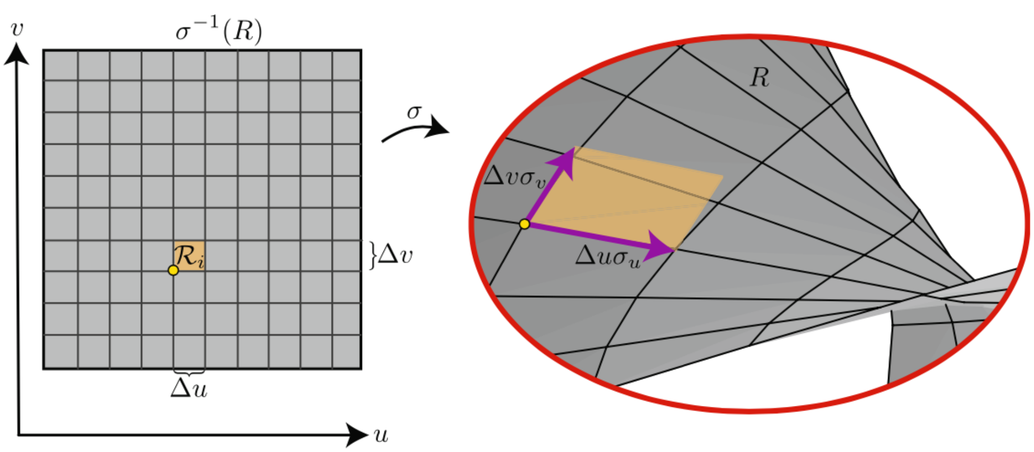
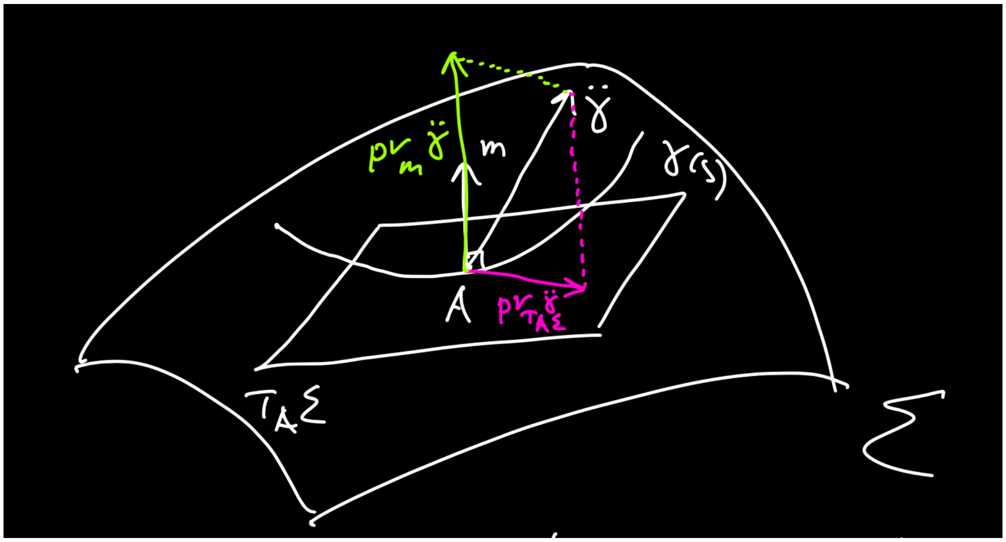


Figure 3.27. $|(\Delta u\sigma_u) \times (\Delta v\sigma_v)|$ approximates the area of $\sigma(\mathcal{R}_i)$

Aside. The determinant of the first fundamental form cannot be negative. First, the scalar product is positive definite by definition (it is an inner product). Also, by Sylvester's criterion, we know that the quadratic form is positive definite if and only if all principal minors are positive definite. Since the first fundamental form a restriction of the scalar product, it is positive definite for a regular surface. Thus, its determinant must be positive.

2.5 Second Fundamental Form

Let Σ be a surface, A a point on the surface, and $T_A\Sigma$ the tangent plane to the surface at A. Since $\Sigma \subseteq \mathbb{R}^3$, there is only one normal line to the plane. So, let m be a unit normal vector at A. It is important to recognize that m is distinct from n, which is the normal vector to the curve at A. Further, $\gamma(s)$ is a curve lying on Σ and s is a natural parameter on γ, and γ passes through A.



We have $k = |\ddot{\gamma}|$ as usual. Also, $\text{pr}_{T_A\Sigma}(\ddot{\gamma})$ is the orthogonal projection of $\ddot{\gamma}$ onto $T_A\Sigma$ and $\text{pr}_m\ddot{\gamma}$ is the normal projection $\ddot{\gamma}$. It follows that

$$\ddot{\gamma} = \text{pr}_{T_A\Sigma}\ddot{\gamma} + \text{pr}_m\ddot{\gamma}$$

since $T_A\Sigma \oplus \text{span}\{m\} = \mathbb{R}^3$. While curvature is the length of the acceleration vector, we would likely to separately consider the lengths of the vectors of each of these projections.

Definition. Normal and Geodesic Curvature

A normal curvature of γ at A is given by

$$k_n = |\text{pr}_m\ddot{\gamma}|$$

A geodesic curvature of γ at A is given by

$$k_g = \left|\text{pr}_{T_A\Sigma}\ddot{\gamma}\right|$$

Proposition. We have

$$k = \sqrt{k_n^2 + k_g^2}$$

This follows simply from the Pythagorean theorem. Thus, if we have the normal and geodesic curvature, we can calculate the total curvature (this means that same as curvature).

Remark. We may compute

$$k_n = |\mathrm{pr}_m \ddot{\gamma}| = (\ddot{\gamma}, m)$$

However, if the angle between $\ddot{\gamma}$ and m passes into a certain range, then we must have $k_n = -(\ddot{\gamma}, m)$. In any case, we have $\pm k_n = (\ddot{\gamma}, m)$ Recall that we have previously computed

$$\dot{\gamma} = r_u \dot{u} + r_v \dot{v}$$

Taking the derivative again, and using product and chain rule, we have

$$\ddot{\gamma} = (r_{uu}\dot{u} + r_{uv}\dot{v})\dot{u} + r_u \ddot{u} + (r_{vu}\dot{u} + r_{vv}\dot{v})\dot{v} + r_v \ddot{v}$$

As a reminder, we have $\gamma = r(f(t))$ where $f : \mathbb{R} \to \mathbb{R}^2$ is defined by $t \mapsto (u(t), v(t))$ and $\frac{d\gamma}{dt} = Dr \circ Df = \begin{pmatrix} r_u & r_v \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}$. Recall also that $r_u, r_v \in T_A\Sigma$, which means that $r_u, r_v \perp m$ and hence $(r_u, m) = 0 = (r_v, m)$. Plugging our expression for $\ddot{\gamma}$ into k_n, we see that

$$\pm k_n = (\ddot{\gamma}, m) = (r_{uu}, m)\dot{u}^2 + 2(r_{uv}, m)\dot{u}\dot{v} + (r_{vv}, m)\dot{v}^2$$

We remark that this is a quadratic form applied to $\dot{u}$ and $\dot{v}$. If we define $L = (r_{uu}, m)$, $M = (r_{uv}, m)$ and $N = (r_{vv}, m)$, then we have

$$\pm k_n = \begin{pmatrix} \dot{u} & \dot{v} \end{pmatrix} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}$$

Definition. Second Fundamental Form

A quadratic form such that its coefficients in the basis r_u, r_v are $\left(\begin{smallmatrix} L & M \\ M & N \end{smallmatrix}\right)$ is called the second fundamental (quadratic) form. The second fundamental form is usually denoted by the roman letters II. Since $\dot{r} = r_u \dot{u} + r_v \dot{v}$, this means that the coordinates of $\dot{r}$ in the basis r_u, r_v are $(\dot{u}, \dot{v})$.

Proposition. If s is a natural parameter, then

$$\pm k_n = \mathrm{II}(\dot{\gamma}, \dot{\gamma})$$

This follows immediately from our definition. The basis is r_u, r_v and the inputs to second fundamental form are in the tangent plane.

Proposition. Let t be an arbitrary parameter on a curve. Then

$$\pm k_n = \frac{\mathrm{II}\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right)}{\mathrm{I}\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right)}$$

Proof. Let t be an arbitrary parameter. Then $\dot{\gamma} = \frac{dr}{ds} = \frac{dr}{dt}\frac{dt}{ds}$ Now, consider that

$$\begin{aligned}
\pm k_n = \mathrm{II}\left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds}\right) &= \frac{\mathrm{II}\left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds}\right)}{1} \\
&= \frac{\mathrm{II}\left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds}\right)}{\mathrm{I}\left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds}\right)} \\
&= \frac{\mathrm{II}\left(\frac{d\gamma}{dt}\frac{dt}{ds}, \frac{d\gamma}{dt}\frac{dt}{ds}\right)}{\mathrm{I}\left(\frac{d\gamma}{dt}\frac{dt}{ds}, \frac{d\gamma}{dt}\frac{dt}{ds}\right)} \\
&= \frac{\left(\frac{dt}{ds}\right)^2 \mathrm{II}\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right)}{\left(\frac{dt}{ds}\right)^2 \mathrm{I}\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right)} = \frac{\mathrm{II}\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right)}{\mathrm{I}\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right)}
\end{aligned}$$

□

Proposition. Suppose A is a point on a curve r and ξ is a tangent vector at A. There exists a parameter t such that $\frac{dr}{dt}(A) = \xi$.

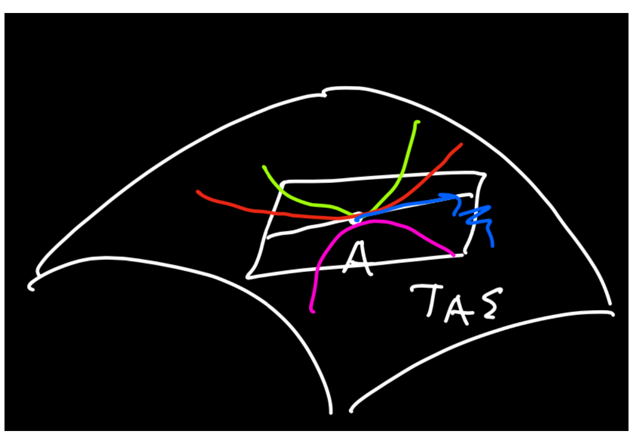
Proof. We can accomplish this if we simply take a natural parameter, and multiply it by the length of ξ. This way we move with constant speed $|\xi|$ along the curve. As a result, the tangent vector at A under such a parametrization will match ξ.

Corollary. The normal curvature only depends on the tangent vector.

$$\pm k_n = \frac{\mathrm{II}(\xi, \xi)}{\mathrm{I}(\xi, \xi)}$$

Proof. We have shown that we may compute the normal curvature from an arbitrary parametrization and that we can obtain an a parametrization given an arbitrary tangent vector at a point on a curve. Thus, given the tangent vector at a point, we can determine the normal curvature.

Corollary. All curves passing through a point A on a surface Σ with the same tangent vector ξ have the same normal curvature, k_n.



This corollary is surprising because from the definition, it appears that normal curvature depends on second-order derivatives (it is the length of a projection of the acceleration

vector onto m). However, the formula for k_n only involves first-order derivatives.
Corollary. Since $k = \sqrt{k_n^2 + k_g^2}$, the curve among these curves (i.e. the curves passing through some point A with the same tangent vector) with the minimal k has $k_g = 0$.
Geodesics, which are curves with geodesic curvature equal to zero, are analogues of straight lines on curved surfaces.

2.6 Linear Algebra and Principal Curvatures

Let A, B be two quadratic forms such that A is positive definite. For our purpose, we may understand a quadratic form as a symmetric, bilinear form.
Theorem. (Sylvester's Law of Inertia). Given a quadratic form Q, there exists a basis β such that $[Q]_\beta$ is a diagonal matrix consisting of ± 1 or 0 on the diagonal.
Theorem. Given a quadratic form, there exists an orthonormal basis for which that form is diagonal.
Proof. A quadratic form may be represented with a symmetric matrix A. Place coefficients in front of the terms x_i^2 in the quadratic form on the diagonal of the matrix and halve coefficients in front of mixed variables, placing them in the remaining locations. Symmetric matrices are diagonalizable over $\mathbb{R}$ as a consequence of the spectral theorem.

□

Theorem. There exists a basis $e_1, \dots, e_n$ such that in this basis the matrix of A has the representation I_n and the matrix B is diagonal, so

$$B = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

Moreover, $\lambda_i \in \mathbb{R}$ are the roots of the equation $\det(B - \lambda A) = 0$ where the matrices B, A are written in any basis. In other words, there exists a basis that simultaneously diagonalizes A and B. Further, to determine the e_i, we note that $(B - \lambda_i A)e_i = 0$.
Proof. By Sylvester's Law of Inertia, there exists a basis $\beta = \{f_1, \dots, f_n\}$ such that $[A]_\beta$ consists of $-1, 0, +1$ on the diagonal. Since A is positive definite, these diagonal entries must all be 1 (by Sylvester's criterion). So, β is a basis for which A is the identity matrix. Using the second theorem, there is an orthonormal basis $\gamma = \{e_1, \dots, e_n\}$ for which B is diagonal. However, since $[A]_\beta$ is the identity matrix, after change of basis from β to γ, it will remain as the identity matrix (i.e. $[A]_\gamma = C^T[A]_\beta C \implies [A]_\gamma = I_n$). Thus, γ satisfies the condition of diagonalizing both quadratic forms in this way.

□

Now, we apply the above theorem to I and II, we know that I is positive definite. From this application we obtain the following theorem.

Theorem. There exists a basis w_1, w_2 (called principal directions) such that in this basis the matrix of the first quadratic form I is $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ and the matrix of the second quadratic form II is

$$\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$$

where λ_1, λ_2 are called principal curvatures.

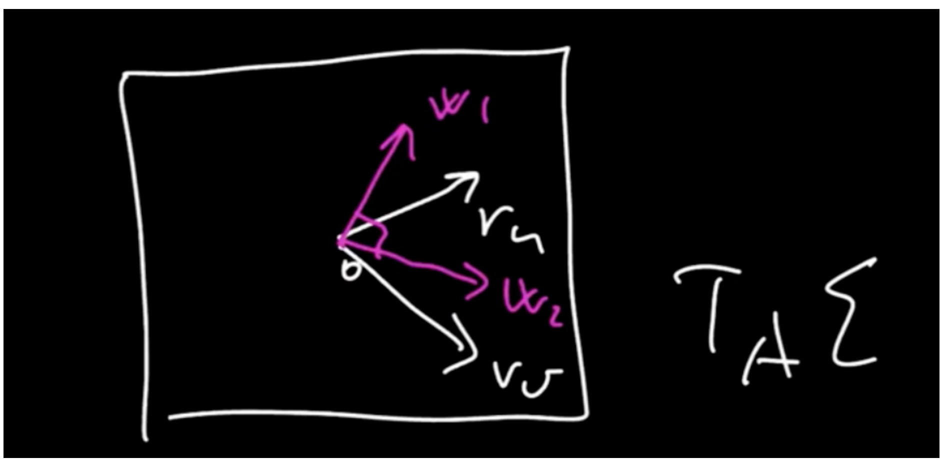
Corollary. The principal directions w_1, w_2 form an orthonormal basis.

Proof. The first fundamental form I in the basis w_1, w_2 has the matrix $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. However, the coefficients of a quadratic form are determined by how the form acts on basis vectors, that is

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\mathrm{I}(w_1,w_1) & \mathrm{I}(w_1,w_2)\\ \mathrm{I}(w_2,w_1) & \mathrm{I}(w_2,w_2)\end{pmatrix} = \begin{pmatrix}(w_1,w_1) & (w_1,w_2)\\ (w_2,w_1) & (w_2,w_2)\end{pmatrix}$$

This shows that w_1 and w_2 are orthogonal, as their scalar product is 0. Further, along the diagonal, we see that the scalar product of w_i with itself is 1, so w_i is a unit vector.

□



Above is a picture of the situation. We have that r_u, r_v are basis vectors that span the tangent plane $T_A\Sigma$ but may not be orthogonal, while the principal directions are orthogonal. However, the principal directions are not unique in that taking the negative direction of w_1 or w_2 would produce an orthonormal basis as well.

Moreover, it may happen that our principal curvatures λ_1, λ_2 are equal. In this case our second quadratic form II will just be a scalar times the identity matrix. If the quadratic form is an identity matrix, the basis may be rotated in any way without changing the matrix. The change of basis matrix between two orthonormal bases is orthogonal, so $Q \mapsto C^TQC$ and $C^TQC = Q$. Thus, in the case where our principal curvatures are equal, any orthonormal basis is a basis of principal directions.

Remark. Notice that there is no canonical order to the principal curvatures, λ_i. How do we address this point that eigenvalues of a matrix are not ordered? We want to consider the symmetric polynomials of these quantities.

Theorem. Any symmetric polynomial may be decomposed into a product and sum of elementary symmetric polynomials.

Aside. Symmetric Polynomials.

1. What is a symmetric polynomial?

A symmetric polynomial is a polynomial $P(x_1, x_2, \ldots, x_n)$ in several variables such that if any of the variables are interchanged, the polynomial remains unchanged.

2.What is an elementary symmetric polynomial?

For n variables $x_1, \ldots, x_n$, there are n elementary symmetric polynomials, each summing up different combinations of variables taken k at at time, for $1 \leq k \leq n$:

$$e_1 = x_1 + x_2 + \ldots + x_n$$
$$e_2 = x_1x_2 + x_2x_3 + \ldots + x_{n-1}x_n$$
$$e_n = \prod_{i=1}^{n} x_i$$

Example. For a decomposition, consider that

$$P(x, y, z) = x^2 + y^2 + z^2 + xy + yz + xz = e_1^2 - e_2$$

Definition. Mean Curvature and Gaussian Curvature

We denote mean curvature by H and Gaussian curvature by K

$$H = \lambda_1 + \lambda_2$$
$$K = \lambda_1\lambda_2$$

Although the sense of mean is average here, we ignore division by 2 in order to simplify formulas which make use of mean curvature, as keeping this coefficient becomes unwieldy. Instead of studying principal curvatures directly, it is more convenient to study mean and Gaussian curvatures.

Remark. From our previous theorem, we know that λ_i are the roots of the characteristic polynomial $\det(\mathrm{II} - \lambda \mathrm{I}) = 0$. We may rewrite this as

$$\det((\mathrm{II} \cdot \mathrm{I}^{-1} - \lambda \mathrm{Id})\mathrm{I}) = 0$$
$$\det(\mathrm{II} \cdot \mathrm{I}^{-1} - \lambda \mathrm{Id}) \cdot \det \mathrm{I} = 0$$
$$\det(\mathrm{II} \cdot \mathrm{I}^{-1} - \lambda \mathrm{Id}) = 0$$

The last equality follows from the positive definiteness of I ensuring that its determinant is nonzero. The last equation shows that λ_i are eigenvalues of $\mathrm{II} \cdot \mathrm{I}^{-1}$. Further, for Gaussian curvature

$$K = \lambda_1 \lambda_2 = \det(\mathrm{II} \cdot \mathrm{I}^{-1}) = \frac{\det \mathrm{II}}{\det \mathrm{I}}$$

To see why $\lambda_1 \lambda_2 = \det(\mathrm{II} \cdot \mathrm{I}^{-1})$, notice that $\mathrm{II} \cdot \mathrm{I}^{-1}$ is diagonal with the diagonal entries as λ_i. On the other hand, for mean curvature we have

$$H = \lambda_1 + \lambda_2 = \mathrm{tr}(\mathrm{II} \cdot \mathrm{I}^{-1})$$

Proposition. One may compute mean and Gaussian curvatures from the first and second fundamental forms

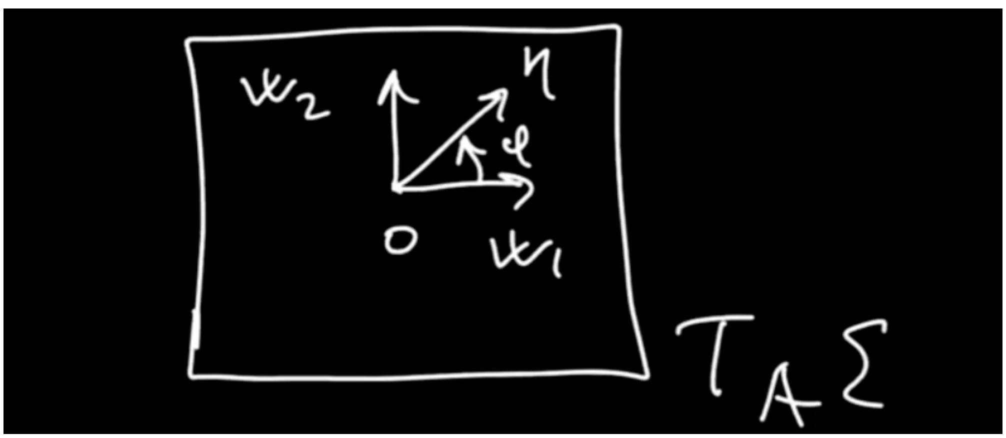
$$K = \frac{\det \mathrm{II}}{\det \mathrm{I}} \text{ and } H = \mathrm{tr}(\mathrm{II} \cdot \mathrm{I}^{-1})$$

The proof is given above.

To find K and H, we do not have to compute principal curvatures. We can find first and second fundamental forms in any basis (i.e. in our usual basis of r_u and r_v). We can do this because the determinant and trace are invariant under change of basis transformations.

2.7 Relating Normal Curvature with Principal Curvature

Let w_1, w_2 be our principal directions and η be a unit tangent vector to a curve.



Given the setup above, the angle φ fully determines η since

$$\eta = w_1 \cos\varphi + w_2 \sin\varphi$$

This means that η has coordinates $(\cos\varphi, \sin\varphi)$ in the basis w_1, w_2. With the same basis, recall that I has the matrix representation $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and II has matrix $\left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)$.

Additionally, normal curvature may be computed as

$$\begin{aligned} \pm k_n &= \frac{\mathrm{II}(\eta,\eta)}{\mathrm{I}(\eta,\eta)} = \mathrm{II}(\eta,\eta) = \begin{pmatrix} \cos\varphi & \sin\varphi \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \\ &= \lambda_1 \cos^2\varphi + \lambda_2 \sin^2\varphi \end{aligned}$$

Theorem. Euler Formula. The proof is stated above.

$$\pm k_n = \lambda_1 \cos^2 \varphi + \lambda_2 \sin^2 \varphi$$

Definition. Oriented Normal Curvature

We ignore the signs associated with normal curvature.

$$\widetilde{k_n} = (\ddot{\gamma}, m)$$

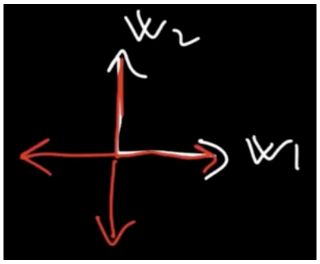
Proposition. We have

$$\widetilde{k_n} = \frac{\mathrm{II}(\xi, \xi)}{\mathrm{I}(\xi, \xi)}$$
$$\widetilde{k_n} = \lambda_1 \cos^2 \varphi + \lambda_2 \sin^2 \varphi$$

Why did we not define oriented normal curvature earlier?
We would like to have a definition suitable for general situations. In a general situation for a k-dimensional surface inside an n-dimensional space, the tangent space will have dimension k and its orthogonal complement will have dimension $n - k$. Thus, there is no unique unit normal vector onto which we may project. The definitions in the proposition only work for hypersurfaces (surfaces of codimension 1).
Remark. $\widetilde{k_n}$ is a function of φ, based on the equation we have stated in the proposition. How do we find the maxima and minima of the Euler formula? We need to determine the critical points and then classify them. Taking the derivative, we have

$$\begin{aligned} \frac{d\widetilde{k_n}}{d\varphi} &= -2\lambda_1 \cos\varphi \sin\varphi + 2\lambda_2 \sin\varphi \cos\varphi \\ &= (\lambda_2 - \lambda_1)\sin(2\varphi) \end{aligned}$$

If $\lambda_1 = \lambda_2$, then $\widetilde{k_n}(\varphi)$ must be a constant function (no maxima and minima). If principal curvatures are equal, then for any direction oriented normal curvature is the same due to the Euler formula. This collaborates with the fact that any orthonormal basis could be taken as principal directions because for a constant function the maximum is the same as the minimum. If $\lambda_1 \neq \lambda_2$, then notice that the derivative is zero only when we have the zeroes of the sine function. Maxima and minima are possible in this case. We know that $\sin 2\varphi \iff 2\varphi = \pi k$ for $k \in \mathbb{Z}$. So $\varphi = \frac{\pi}{2}k$ for integers k. But this means that the extrema occur at the principal directions, or their negatives.



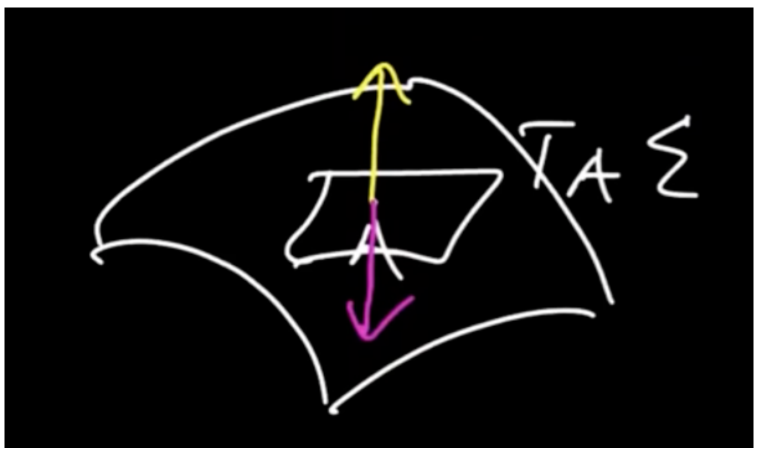
Further, by plugging $\frac{\pi}{2}k$ into Euler's formula, we notice that either the contribution cos or sin will vanish, and only λ_1 or λ_2 remain.

Proposition. The principal curvatures are minimal or maximal normal curvatures of curves going through the given point on a surface. These curvatures are obtained for curves tangent to principal directions.

Remark. The geometric meaning of the above proposition is that if we take all the possible tangent vectors and all the possible curves with these tangent vectors, then we obtain the maximum and minimum normal curvature when the vectors are along the principal directions, and the obtained curvatures are the principal curvatures.

2.8 The Choice of Unit Normal Vector

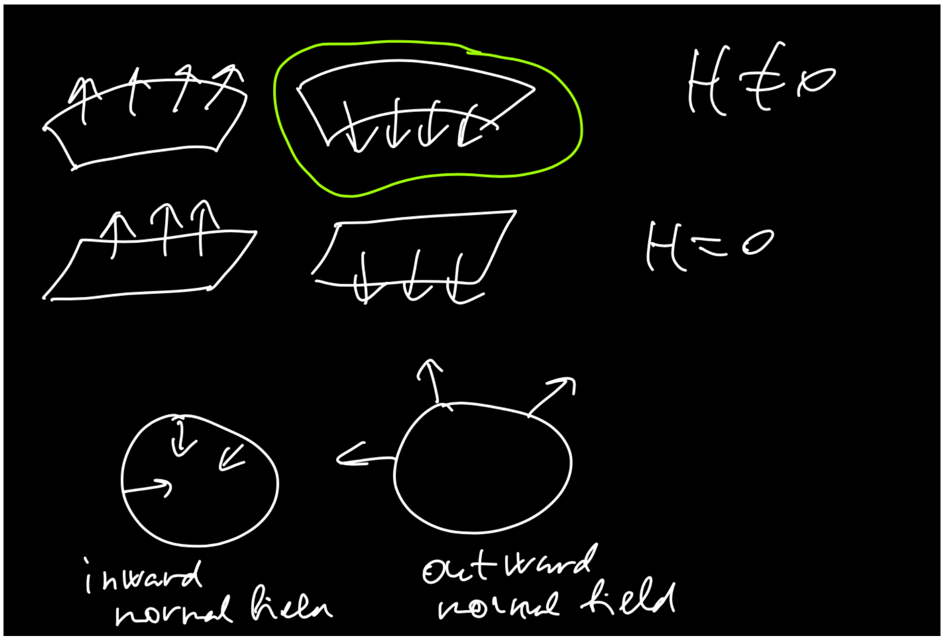
Remark. The normal vector m is not unique. Below, we see that at a point A, we can have a unit normal vector in either direction.



What happens if we change our choice of unit normal vector (i.e. $m \mapsto -m$)?

- The first quadratic form does not change, as it does not depend on the unit normal vector at all: $\mathrm{I} \mapsto \mathrm{I}$.
- Since the second quadratic form is composed of scalar products of the form (r_{**}, m), mapping to $-m$ changes the sign: $\mathrm{II} \to -\mathrm{II}$.
- $\widetilde{k_n} \mapsto -\widetilde{k_n}$, by definition.
- Similarly, we see that $\mathrm{II} \cdot \mathrm{I}^{-1} \mapsto -\mathrm{II} \cdot \mathrm{I}^{-1}$.
- Since the eigenvalues of $\mathrm{II} \cdot \mathrm{I}^{-1}$ are principal curvatures, we see that $\lambda_i \mapsto -\lambda_i$.
- Gaussian curvature does not change $K = \lambda_1\lambda_2 \mapsto K$. On the contrary, mean curvature changes sign $H = \lambda_1 + \lambda_2 \mapsto -H$.

- $Hm \mapsto Hm$, this field is called the **mean curvature normal vector**. Also, $\frac{Hm}{|H|} \mapsto \frac{Hm}{|H|}$ (since m is a unit vector, we omit $|m|$).
  From the point of view of differential geometry, when we have a surface with mean curvature not equal to zero, then there is a preferred choice of unit normal field. This idea is shown below

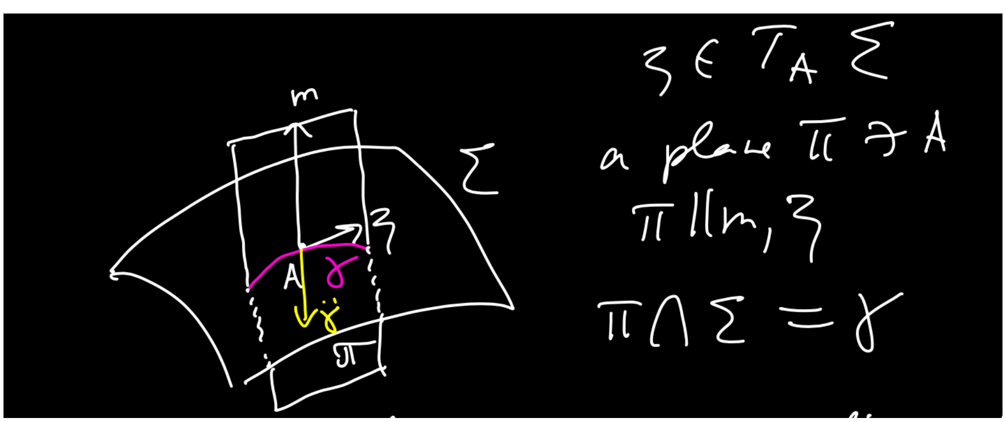


2.9 Differential Geometry Without Normal Curvature

Sometimes in older books, we may come across the treatment of differential geometry without using normal curvature. We expand on how this is accomplished through **normal sections**.

Definition. Normal Section

A curve γ, as shown below, is called a normal section of a surface Σ at a point A in the direction of a tangent vector ξ.



Above, the plane which contains m and intersects the surface is called the normal plane. The normal plane intersects the surface in some curve γ. It may be proven that in a sufficiently small neighborhood around A, the curve γ is smooth and regular.

Remark. Observe that $\xi \subseteq \Pi$. So, γ is a planar curve and $\ddot{\gamma} \subseteq \Pi$, since the derivatives of a planar curve must all belong to the same plane as the curve. If s is a natural parameter, then $\ddot{\gamma} \perp \xi$. It follows that $\ddot{\gamma} \parallel m$. Then

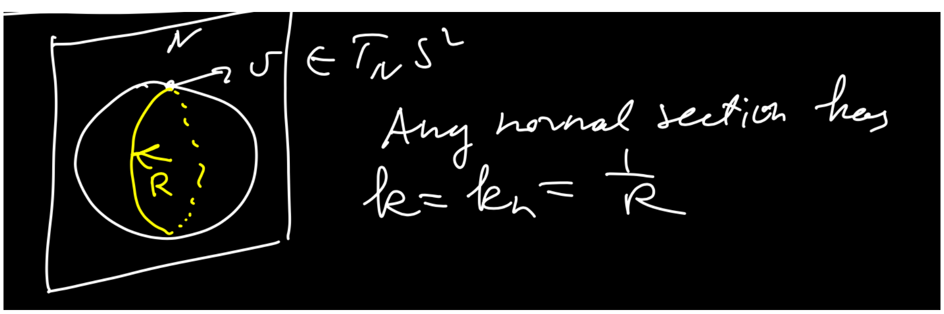
$$\begin{aligned} \mathrm{pr}_{T_A\Sigma}\ddot{\gamma} = 0 &\implies k_g = 0 \\ \mathrm{pr}_m\ddot{\gamma} = \ddot{\gamma} &\implies k_n = k \end{aligned}$$

This shows why normal sections are nice: the normal curvature of a curve in the section coincides with the total curvature.

Proposition. For a normal section, we have $k_n = k$.

Remark. If two curves have the same tangent vector, then they have the same normal curvature. For a curve $\tilde{\gamma}$ its k_n is equal to k_n of the normal section in the direction $\ddot{\tilde{\gamma}}$, that is, k of this normal section. Thus, in classical literature, instead of discussing normal curvature, authors discussed curvature of normal section with the same tangent vector.

When solving problems in differential geometry, it is useful to think about the geometrical meaning first before solving the problem. For instance, suppose we have a sphere with radius R, a tangent vector at the north pole v, and a normal plane which slices the sphere in the direction of this tangent vector, through the north pole.



Since we see that the normal section is a great circle with radius R, we conclude that $k = k_n = \frac{1}{R}$ from what we know about the curvature of circles. By Euler formula, we know that

$$\lambda_1 \cos^2\varphi + \lambda_2 \sin^2\varphi = \frac{1}{R}$$

This means that our principal curvatures are $\lambda_1 = \lambda_2 = \frac{1}{R}$, which shows that our Gaussian curvature is $K = \frac{1}{R^2}$. Notice how we were able to determine this information without explicitly calculating the first or second fundamental forms.

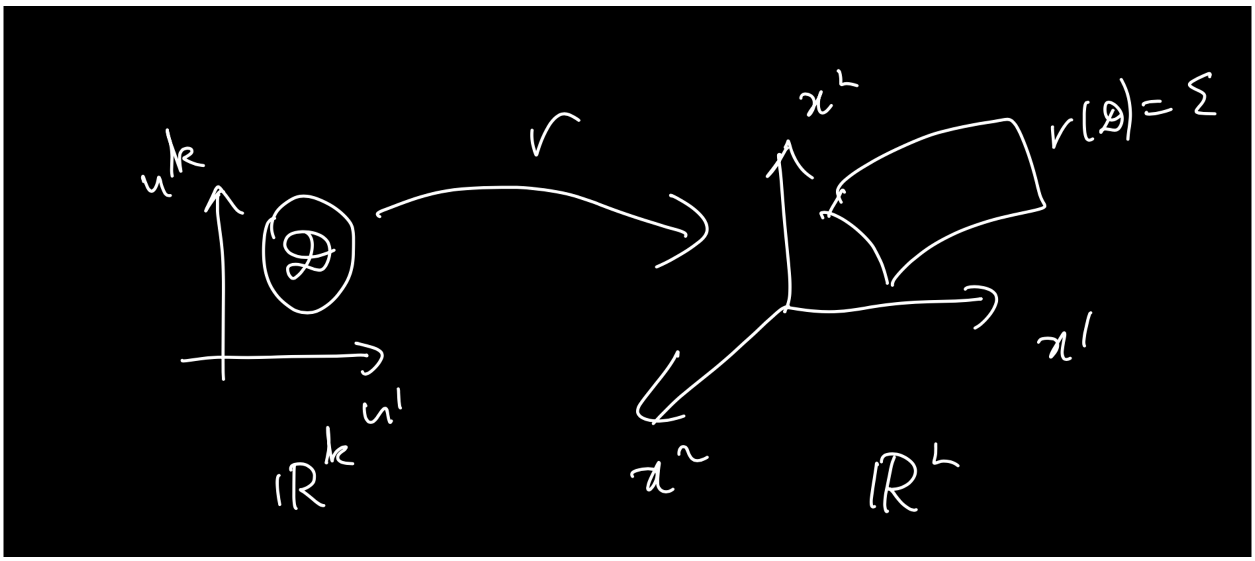
2.10 k-dimensional Surfaces in n-dimensional Euclidean Space

Definition. k-dimensional surface

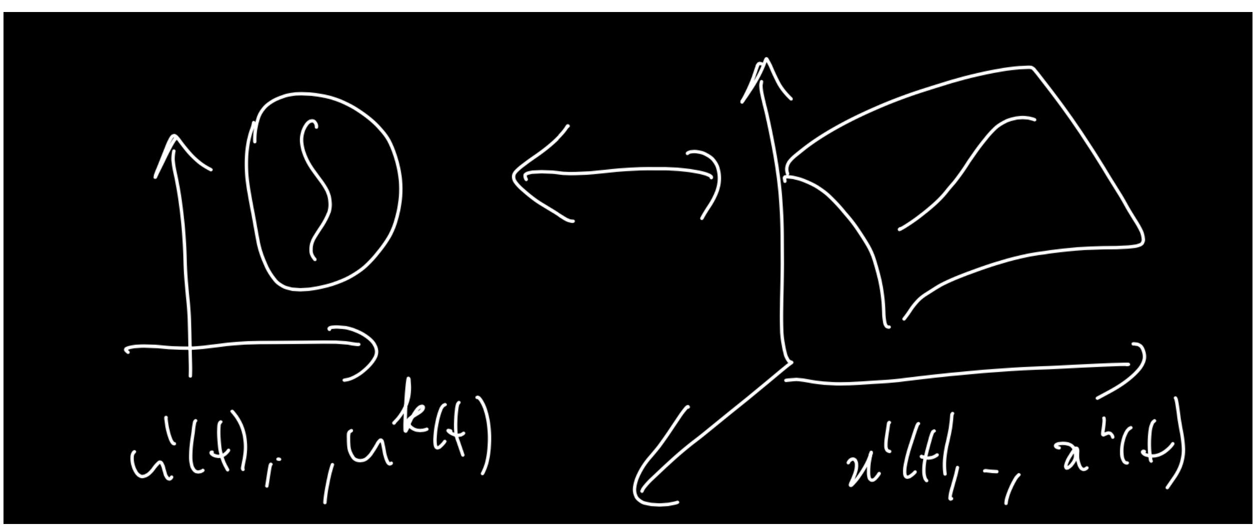
A k-dimensional regular smooth surface in $\mathbb{R}^n$ is a map $r : \mathscr{D} \to \mathbb{R}^n$, where $\mathscr{D} \subseteq \mathbb{R}^k$, such that

1. $r \in \mathscr{C}^\infty(\mathscr{D}, \mathbb{R}^n)$
2. rank $J = k$ where $J = \left(\frac{\partial r^i}{\partial u^j}\right)$

Below is a drawing of the setup for a general surface.



The coordinates $u_1, \ldots, u_k$ are local coordinates and $x_1, \ldots, x_n$ are called global coordinates in the ambient space. The analogous local and global representation of curves lying on the surface holds in this context as well.



Similarly, taking the derivative of $r(u_1(t), \ldots, u_k(t))$, we have

$$\dot{r} = \sum_{i=1}^{k} r_{u_i} \dot{u}_i$$

where $r_{u_i} = \frac{\partial r}{\partial u_i}$. This equation just follows from the chain rule. So, any velocity vector is a linear combination of the partial derivatives with respect to one of the input variables. Given a point $A \in \Sigma$, the tangent space is defined by

$$T_A\Sigma = \operatorname{span}(r_{u_1}(A), \ldots, r_{u_k}(A))$$

This is called the tangent space to Σ at a point A.

Definition. Normal Space

The normal space to Σ at a point A is the orthogonal complement of the tangent space to the surface at that point

$$N_A\Sigma = (T_A\Sigma)^{\perp}$$

Proposition. The dimension of the tangent space is k and the dimension of the normal space (orthogonal complement) is $n - k$

$$\dim(T_A\Sigma) = k \text{ and } \dim(N_A\Sigma) = n - k$$

We know this since the vectors $r_{u_1}, \ldots, r_{u_k}$ are linearly independent. This is equivalent to the condition that the Jacobian matrix has rank k. The main difficulty in our generalization to k-dimensional surfaces is that the normal space is no longer of dimension 1. With this setup, we also have that

$$T_A\Sigma \oplus N_A\Sigma = \mathbb{R}^n \text{ and } T_A\Sigma \cap N_A\Sigma = \{0\}$$

The definition of the first fundamental form remains the same as the restriction of the scalar product to the tangent space:

$$I_A = \langle \cdot, \cdot \rangle \Big|_{T_A\Sigma}$$

Remark. In the basis $r_{u_1}, \ldots, r_{u_k}$ the matrix of I is

$$\begin{pmatrix} \langle r_{u_1}, r_{u_1} \rangle & \cdots & \langle r_{u_1}, r_{u_k} \rangle \\ \vdots & \ddots & \vdots \\ \langle r_{u_k}, r_{u_1} \rangle & \cdots & \langle r_{u_k}, r_{u_k} \rangle \end{pmatrix}$$

Although we can define the first fundamental form in the same way, because the normal space is no longer 1-dimensional, we cannot define the second fundamental form in the same

way. We need another approach to handle the second fundamental form, II.
Suppose we have a point A and a vector V beginning at that point. Let $U \subseteq \mathbb{R}^n$ be a neighborhood of A and $f : U \to \mathbb{R}^n$. The dimensions of the domain and codomain of f do not always have to match.

Definition. Directional Derivative

The directional derivative is a derivative of f at A along a vector V, given by

$$\partial_V f(A) = \lim_{\varepsilon \to 0} \frac{f(A + \varepsilon V) - f(A)}{\varepsilon}$$

Proposition. If $V = (V^1, \ldots, V^n)$ in the standard basis of $\mathbb{R}^n$, then

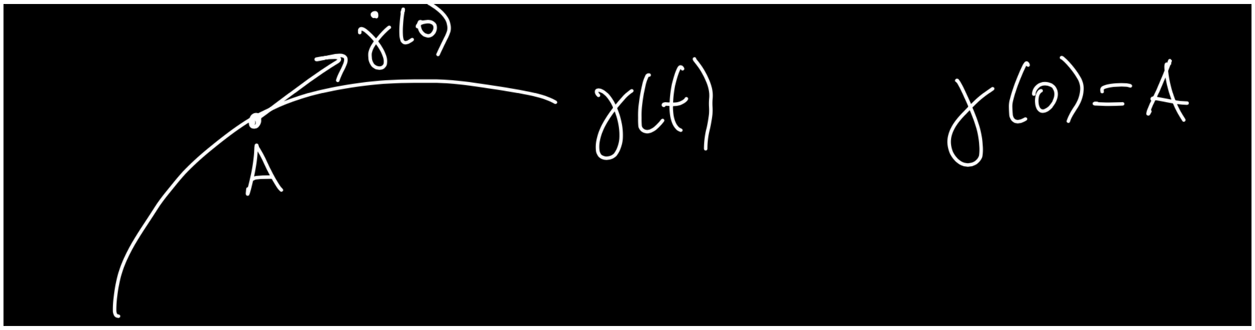
$$\partial_V f(A) = \sum_i \frac{\partial f}{\partial x^i}(A) \cdot V^i$$

Proof. We can think of this as an application of the chain rule when taking the derivative with respect to ε

$$\begin{aligned} \partial_V f(A) &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} f(A + \varepsilon V) = \partial f(A) \circ (A + \varepsilon V)'(0) \\ &= \sum_i \frac{\partial f}{\partial x^i}(A) \cdot V^i \end{aligned}$$

□

Consider a curve $\gamma(t)$ such that $\gamma(0) = A$, along with its tangent vector at that point $\dot{\gamma}(0)$.



Proposition. The following equation holds.

$$\boxed{\partial_{\dot{\gamma}(0)} f(A) = \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t))}$$

In other words, taking the derivative of f along the direction of the velocity vector of a curve at a point A is the same as considering the function along the points of the curve and then taking the derivative with respect to the parameter of the curve, t, then evaluating at the point where the curve travels through A.

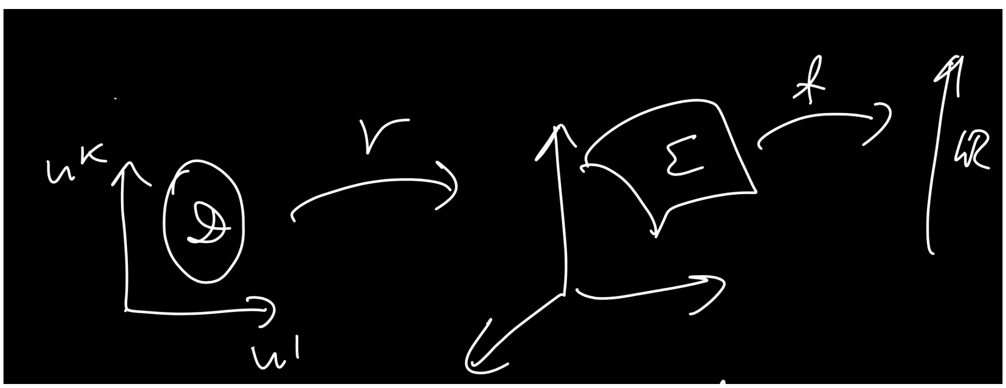
Proof. Let $\gamma(t) = (x^1(t), x^2(t), \dots, x^n(t))$ and $\dot{\gamma}(t) = (\dot{x}^1(t), \dot{x}^2(t), \dots, \dot{x}^n(t))$, these are indices and not exponents. From the previous proposition, we know that

$$\begin{aligned}\partial_{\dot{\gamma}(0)} f(A) &= \sum_i \frac{\partial f}{\partial x^i}(A) \cdot \frac{dx^i}{dt}(0) \\ &= \frac{d}{dt}\bigg|_{t=0} f(x^1(t), \dots, x^n(t)) \\ &= \frac{d}{dt}\bigg|_{t=0} f(\gamma(t))\end{aligned}$$

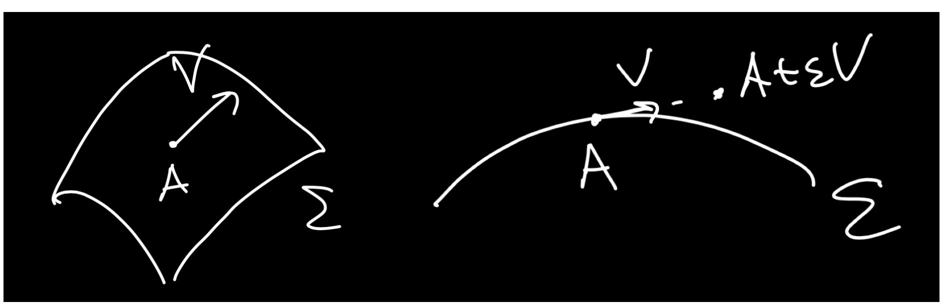
where the penultimate equality follows by the chain rule. □

This is a very important proposition to remember. The point is taking the directional derivative in the direction of the velocity vector at a point is the same as restricting the surface to a curve, taking the derivative, and evaluating at the point.

Let $\Sigma \subseteq \mathbb{R}^n$ be a surface and let $f : \Sigma \to \mathbb{R}$. So our setup looks like



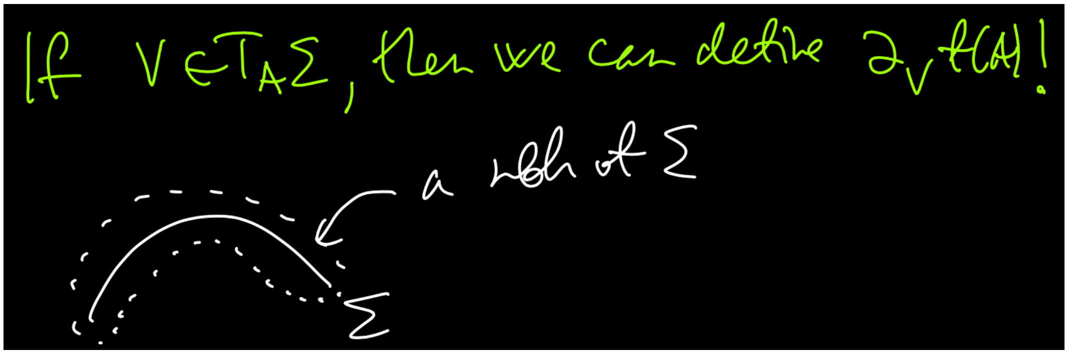
Let $f(u^1, \dots, u^k)$ be a smooth function. This is shorthand for $(f \circ r)(u^1, \dots, u^k)$.



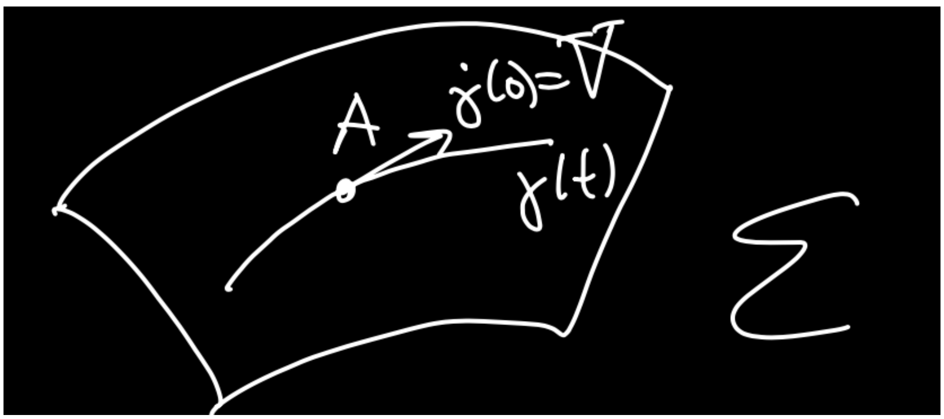
How do we define $\partial_V f(A)$? From our previous definition, we know that

$$\partial_V f(A) = \lim_{\varepsilon \to 0} \frac{f(A + \varepsilon V) - f(A)}{\varepsilon}$$

Remark. If $f(A + \varepsilon V) \notin \Sigma$, then this value of the function is not defined. This creates an issue computing the directional derivative. However, if $V \in T_A\Sigma$, then we can define $\partial_V f(A)$!



Let us extend f in a neighborhood of Σ. Then $f(A + \varepsilon V)$ and $\partial_V f(A)$ are defined. Recall that tangent vectors are defined as velocity vectors of curves. Thus, given a tangent vector $V \in T_A\Sigma$, there exists a curve $\gamma(t)$ on Σ such that $\gamma(0) = A$ and $\dot{\gamma}(0) = V$.



Remark. Recall from our proposition that

$$\partial_V f(A) = \frac{d}{dt}\bigg|_{t=0} f(\gamma(t))$$

This shows that $\partial_V f(A)$ is defined and does not depend on the way in which we extend f in the neighborhood. To be explicit, since $\gamma(t) \in \Sigma$, we know that $f(\gamma(t))$ is defined and $\frac{d}{dt}\big|_{t=0} f(\gamma(t))$ is defined and does not depend on an extension of f from Σ to a neighborhood of Σ. Hence. $\partial_V f(A)$ is well-defined.

How can we extend f from Σ to a neighborhood of Σ

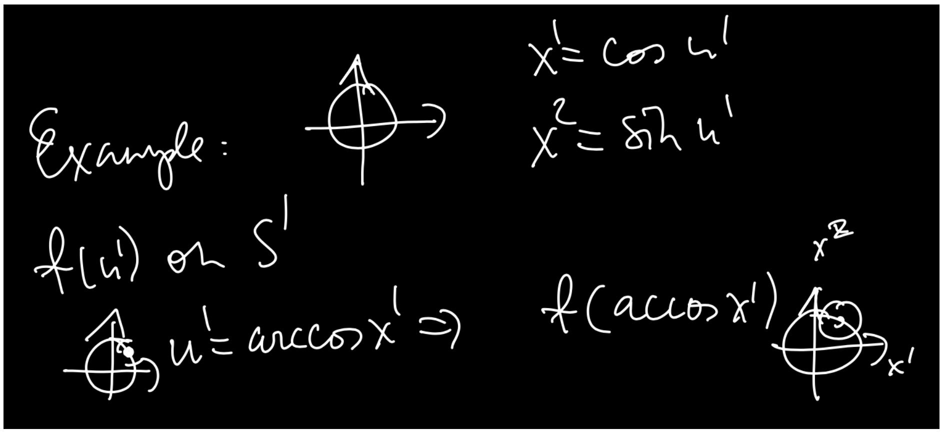
There are many ways in which we may extend. Recall that in a neighborhood of any point of Σ, we may use the inverse function theorem to express local coordinates in terms of some global coordinates. For instance, by the inverse function theorem we have

$$\begin{aligned} u^1 &= u^1(x^1, \dots, x^k) \\ &\vdots \\ u^k &= u^k(x^1, \dots, x^k) \end{aligned}$$

Thus, $f(u^1(x^1,\ldots,x^k),\ldots,u^k(x^1,\ldots,x^k))$ is defined in a neighborhood.
As a concrete example, consider a circle with the parametrization

$$x^1 = \cos u^1 \text{ and } x^2 = \sin u^1$$

Here, superscripts indicate indices and not exponents. Additionally, suppose we have a smooth function $f(u^1)$ on S^1. Then, we see that $u^1 = \arccos x^1$ in some neighborhood of the circle and so $f(\arccos x^1)$ is an extension. However, notice that $f(\arcsin x^2)$ is another extension.



Here, u^1 is the local coordinate. When we consider $u^1 = \arccos x^1$, the second coordinate becomes $\sin(u^1) = \sin(\arccos x^1)$. So the meaning is actually

$$(f \circ r)(\arccos x^1, \sin(\arccos x^1)) = f(\arccos(x^1))$$

Consider the drawing of the circle on the bottom right. We call $f(\arccos x^1)$ an extension because it is valid for values of x^1 that do not lie solely on the circle, but in some neighborhood of the point.

Consider $\gamma(t)$ in local coordinates

$$\gamma(t) = (u^1(t),\ldots,u^k(t))$$

Then in $\mathbb{R}^n$, our curve will be given by

$$\gamma(t) = r(u^1(t),\ldots,u^k(t))$$
$$\dot\gamma(0) = r_{u^1}\dot u^1(0) + \ldots + r_{u^k}\dot u^k(0)$$

This means that $V = \dot\gamma(0)$ has coordinates $(\dot u^1(0),\ldots,\dot u^k(0))$ in the basis $r_{u^1},\ldots,r_{u^k}$.
Then

$$\begin{aligned}\partial_V f(A) &= \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) \\ &= \sum_i \frac{\partial f}{\partial u^i}(A)\cdot \dot u^i(0) = \sum_i \frac{\partial f}{\partial u^i}(A)\cdot V^i\end{aligned}$$

Proposition. If $u^1, \ldots, u^k$ are local coordinates and $V \in T_A\Sigma$ has coordinates $(V^1, \ldots, V^k)$ in the basis $r_{u^1}(A), \ldots, r_{u^k}(A)$, then

$$\partial_V f(A) = \sum_{i=1}^{k} \frac{\partial f}{\partial u^i}(A) \cdot V^i$$

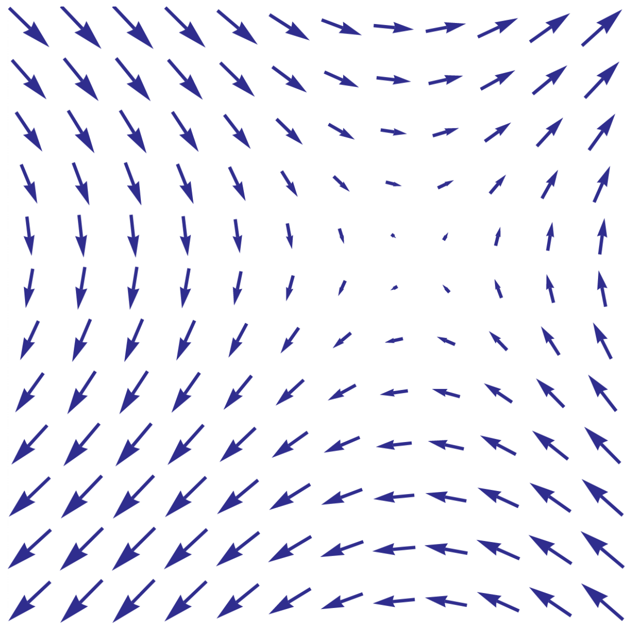
Notice the similarity of this proposition to the proposition following the definition of directional derivative, where the difference is in the basis.

Q: Why does the basis differ? The basis differs because this proposition is a version of the previous proposition involving local coordinates rather than global ones. We can choose which basis we are in as long as we are consistent with how matrices are composed.

3 The Covariant Derivative

3.1 Vector Fields

We now turn to vector fields, which we may think of as a function $\vec{Y} : U \to \mathbb{R}^n$, for $U \subseteq \mathbb{R}^n$. Intuitively, the idea is that each vector depends on a point in the space. Below is an example of a vector field.



Recall the definition of directional derivative. Notice that this definition consists only of linear operations, so replacing f with $\vec{Y}$ has meaning.

Definition. Directional Derivative of a Vector Field

A derivative of a vector field $\vec{Y}$ at a point A along a vector V is given by

$$\partial_V Y(A) = \lim_{\varepsilon \to 0} \frac{Y(A + \varepsilon V) - Y(A)}{\varepsilon}$$

In the case where V is instead $\vec{X}$, another vector field, then it is interpreted as $\partial_{\vec{X}(A)} \vec{Y}(A)$.

Remark. If we write Y in **the standard basis**, then operations in the above definition are performed coordinate-wise. So, if we identify Y with its coordinates in the standard basis

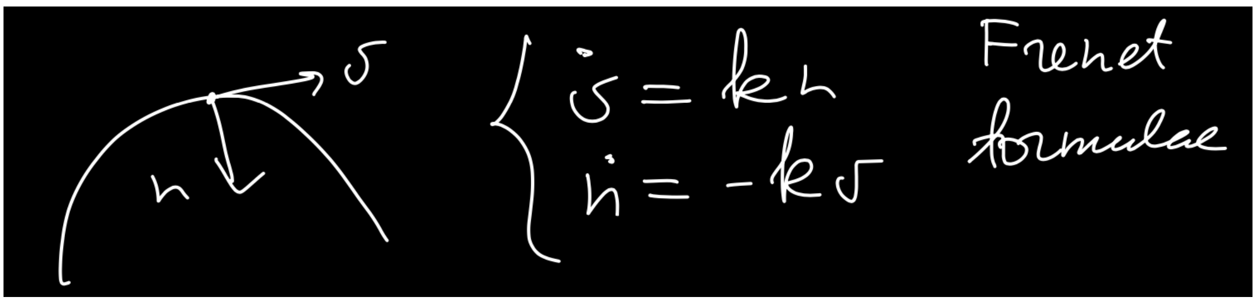
$\begin{pmatrix} Y^1 \\ \vdots \\ Y^n \end{pmatrix}$, then its derivative is

$$\partial_V Y(A) = \begin{pmatrix} \partial_V Y^1(A) \\ \vdots \\ \partial_V Y^n(A) \end{pmatrix}$$

Notice that everything we discussed before regarding the scalar function f on surfaces also applies to vector fields Y. As before, if Y is a vector field on a surface Σ and $V \in T_A\Sigma$, then $\partial_V Y(A)$ is defined.

What is the purpose of studying all the previous concepts?

Recall the Frenet formulae and its setup, as below.



Observe that v, n are vector fields. In particular, v is a tangent vector field. By our previous proposition, we may rewrite our formulas as

$$\partial_v v = kn$$
$$\partial_v n = -kv$$

To be explicit, using the proposition gives us

$$\partial_{v(A)} v(A) = \frac{d}{ds}\bigg|_{s=0} v(\gamma(s)) = \frac{d}{ds}\bigg|_{s=0} \dot{\gamma}(s) = kN(0)$$
$$\partial_v n(A) = \frac{d}{ds}\bigg|_{s=0} n(\gamma(s)) = \frac{d}{ds}\bigg|_{s=0} N(s) = -kv$$

where γ represents our curve and $\gamma(0) = A$. It appears that Frenet formulas are about derivatives of vector fields along another vector field. Notice that at each point, v is tangent and n is normal. Yet surprisingly, the derivative of a tangent vector field is not tangent and the derivative of a normal vector field is not normal. Thus, this shows us that when we take derivative of a tangent vector field, tangency or normality are not preserved.

Our goal is to write down analogues of the Frenet formulae for an arbitrary k-dimensional surface in $\mathbb{R}^n$.

Definition. Tangent Vector Field

A vector field Y on a k-dimensional surface Σ is tangent if for all $A \in \Sigma$

$$Y(A) \in T_A\Sigma$$

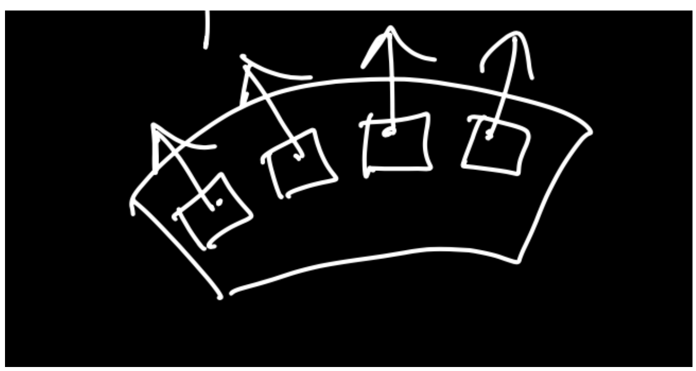
This means that every point of the vector field maps to a vector in the tangent space at that point of the surface.

Definition. Normal Vector Field

A vector field η on Σ is called normal if for all $A \in \Sigma$

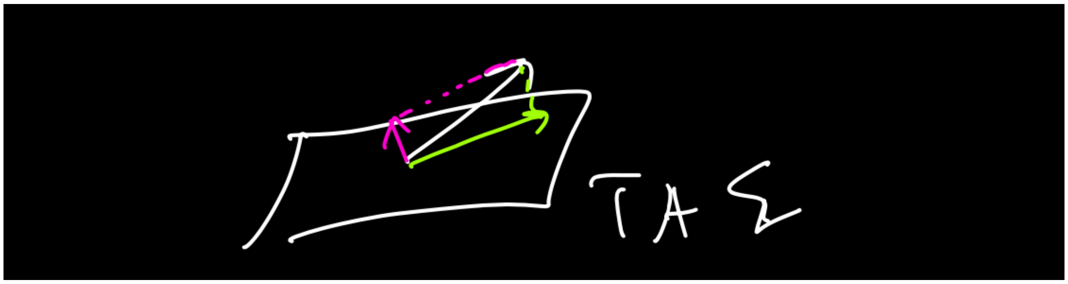
$$\eta(A) \in N_A\Sigma \iff \eta(A) \perp T_A\Sigma$$

that is, $\eta(A)$ belongs to the normal space.



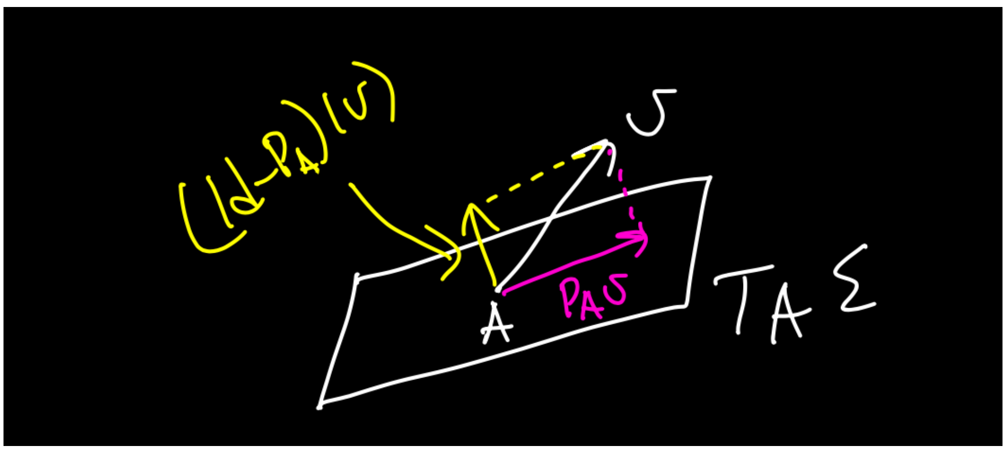
These vector fields are not unique.

Remark. An arbitrary vector field on Σ is not necessarily tangent or normal. However, given some point on this arbitrary vector field, the vector that it maps to may be split into components belonging to the tangent space and normal space.



Let $P_A : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projector $T_A\Sigma$. The orthogonal projector is unique. From linear algebra, there is a one-to-one correspondence between linear subspaces of Euclidean space and orthogonal projectors onto it.

Remark. $\mathrm{Id} - P_A$ is the orthogonal projector on $N_A\Sigma$. The situation is as shown below:



Notice that since our tangent vectors depend on the point A, this means our orthogonal projectors also depend on that point. Thus, we call P the field of orthogonal projectors. In this context, field means that at each point in space, there exists an operator (the orthogonal projector).

Exercise. P_A depends on A smoothly.

3.2 Generalizing the Frenet Formulae

Let Σ be a k-dimensional surface, and $A \in \Sigma$. Let $X \in T_A\Sigma$ and let Y be a tangent vector field on Σ. Let η be a normal vector field on Σ.

$$\partial_X Y(A) = \underbrace{P_A(\partial_X Y(A))}_{\nabla_X Y} + \underbrace{(\mathrm{Id} - P_A)(\partial_X Y(A))}_{B(X,Y)}$$

$$\partial_X \eta(A) = P_A(\partial_X \eta(A)) + (\mathrm{Id} - P_A)(\partial_X \eta(A))$$

Definition. $B(X,Y)$

We call

$$B(X,Y) = (\mathrm{Id} - P_A)(\partial_X Y(A))$$

the **second quadratic (fundamental) form**.

Recall that $T_A\Sigma$ is the tangent space to Σ at A.

Definition. $\Gamma(T\Sigma)$, $\Gamma(N\Sigma)$

We denote by $\Gamma(T\Sigma)$ the space of tangent vector fields on Σ.
We denote by $\Gamma(N\Sigma)$ the space of normal vector fields on Σ.

Studying the object B in greater detail, since X is a vector in the tangent space and Y is a tangent vector field, we see that

$$B : T_A\Sigma \times \Gamma(T\Sigma) \to N_A\Sigma$$

Proposition. $B(X,Y)$ is linear with respect to X, its first argument.

Proof. Observe that

$$\partial_X f(A) = \sum_i \frac{\partial f}{\partial x^i}(A) \cdot X^i$$

is linear with respect to X (composition with the Jacobian matrix is linear). Why is it f here instead of Y? Further, P_A is a linear operator. The sum and composition of linear operators is linear, so we are done.

□

Let now X be a tangent vector field. Then $B(X,Y) = (\mathrm{Id} - P)(\partial_X Y)$ could be evaluated at each point on the surface. In this case, we have

$$B : \Gamma(T\Sigma) \times \Gamma(T\Sigma) \to \Gamma(N\Sigma)$$

Remark. $B(X,Y)(A)$ depends on $X(A)$ and Y in a neighborhood of A.

Exercise. There exists a vector field $[X,Y]$, the commutator of X and Y, such that $\forall f$

$$\partial_X(\partial_Y f) - \partial_Y(\partial_X f) = \partial_{[X,Y]} f$$

Definition. Commutator

$[X,Y]$ is called a commutator of vector fields X and Y.

Exercise 2. $\partial_X Y - \partial_Y X = [X,Y]$

Exercise 3. The commutator of two tangent vector fields is a tangent vector field.

Proposition. Let X, Y be two tangent vector fields. Then

$$B(X,Y) = B(Y,X)$$

Proof. Notice that

$$\begin{aligned} B(X,Y) - B(Y,X) &= (\mathrm{Id} - P)(\partial_X Y) - (\mathrm{Id} - P)(\partial_Y X) \\ &= (\mathrm{Id} - P)(\partial_X Y - \partial_Y X) \\ &= \underbrace{(\mathrm{Id} - P)}_{\text{normal projector}} \underbrace{([X,Y])}_{\text{tangent}} = 0 \end{aligned}$$

Corollary. 1. $B(X,Y)$ is linear with respect to Y.
Proof. $B(X,Y) = B(Y,X)$ and we know that B is linear with respect to the first argument □

2. $B(X,Y)(A)$ depends only on $X(A)$ and $Y(A)$.
As a result $B : T_A\Sigma \times T_A\Sigma \to N_A\Sigma$ is bilinear and symmetric. Q: Is Part 2 of the Corollary related to the symmetry or bilinearity at all? We see that B is a symmetric, bilinear form, but that it is vector-valued. It is surprising that in the second fundamental form is vector-valued in the multidimensional case.
Exercise. Let Σ be a 2-dimensional surface in the euclidean $\mathbb{R}^3$. Let m be a unit normal vector field. Then

$$B(X,Y) = \mathrm{II}(X,Y) \cdot m$$

This explains how $B(X,Y)$ is a generalization of the second fundamental form we have seen earlier.
Questions
1. For this proposition, does it depend on the parametrization?

3.3 Connection

Let Y be a tangent vector field and let X be a tangent vector or a tangent vector field. Then, $\partial_X Y$ is a vector or a vector field, respectively. Recall

$$\partial_X Y = \underbrace{P(\partial_X Y)}_{\nabla_X Y} + \underbrace{(\mathrm{Id} - P)(\partial_X Y)}_{B(X,Y)}$$

Previously, we established that $B : T_A\Sigma \times T_A\Sigma \to N_A\Sigma$ is

1. Bilinear
2. Symmetric

The main difference for this version of the second fundamental form is that it outputs a vector rather than a number.

Definition. Covariant Derivative

The function $\nabla : T_A\Sigma \times \Gamma(T\Sigma) \to T_A\Sigma$ or $\nabla : \Gamma(T\Sigma) \times \Gamma(T\Sigma) \to \Gamma(T\Sigma)$

$$\nabla_X Y = P(\partial_X Y)$$

is called a covariant derivative of a tangent vector field Y along (or w.r.t.) a tangent vector (or tangent vector field) X.

∇ is often called an (affine) connection. This term 'connection' has no relation to the idea of connection in topology. The idea of connection is important in differential geometry because it manifests in many unexpected, but equivalent forms (covariant derivative, parallel transport, distribution of hypersurfaces of a special type, etc).

Properties of ∇

1. $$\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y$$
   where $f_i \in C^\infty(\Sigma)$.
2. $$\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$$
3. $$\nabla_X (fY) = \partial_X f \cdot Y + f \cdot \nabla_X Y$$
   where $f \in C^\infty(\Sigma)$. This is the product rule.
   In more complex settings, these three properties are taken as axioms of a connection.
4. Symmetricity.
   $$\nabla_X Y - \nabla_Y X = [X, Y]$$
5. $$\partial_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$$
   where $\langle \cdot, \cdot \rangle$ indicates the scalar product of tangent vector fields. There are several ways to refer to this property. We may call ∇ a metric connection or say ∇ is compatible with the scalar product.

Proof.

1. $\partial_X Y$ is linear with respect to X and P is a linear operator. The composition of linear functions is linear.

2. The derivative of a sum is the sum of the derivatives. So, it follows from the linearity of ∂_X.

3. Observe that

$$\begin{aligned}\nabla_X(fY) &= P(\partial_X(fY)) \\ &= P\left(\sum_i \frac{\partial(fY)}{\partial u^i} X^i\right) \\ &= P(\partial_X f \cdot Y + f \cdot \partial_X Y) \\ &= \partial_X f \underbrace{P(Y)}_{P(Y)=Y} + fP(\partial_X Y) \quad = \partial_X f \cdot Y + f \cdot \nabla_X Y\end{aligned}$$

4. Observe that

$$\begin{aligned}\nabla_X Y - \nabla_Y X = P(\partial_X Y) - P(\partial_Y X) &= P(\partial_X Y - \partial_Y X) \\ &= P([X,Y]) = [X,Y]\end{aligned}$$

5. Notice that $\partial_X \langle Y, Z\rangle = \langle \partial_X Y, Z\rangle + \langle Y, \partial_X Z\rangle$. To see this, consider that in global coordinates, we have

$$\begin{aligned}\partial_X\left(\sum_i Y^i Z^i\right) &= \sum_i (\partial_X Y^i) Z^i + \sum_i Y^i(\partial_X Z^i) \\ &= \langle \nabla_X Y + B(X,Y), Z\rangle + \langle Y, \nabla_X Z + B(X,Z)\rangle \\ &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle\end{aligned}$$

where the last equality follows since $B(X,Y)$ and Z are orthogonal, and similarly for $B(X,Z)$ and Y.

■

Objects in geometry satisfying these five properties are called Levi-Civita connections.

Definition. Gauss Derivational Equation

The Gauss Derivational Equation is given by

$$\partial_X Y = \nabla_X Y + B(X,Y)$$

Further, recall that its particular case is the first Frenet equation.

$$\partial_v v = kn$$

This means that $\nabla_v v = 0$ and $B(v,v) = kn$.
Let η be a normal vector field and X be a tangent vector or tangent vector field. We have

$$\partial_X \eta = \underbrace{P(\partial_X \eta)}_{-W_\eta(X)} + \underbrace{(\mathrm{Id} - P)(\partial_X \eta)}_{\nabla_X^{N\Sigma}\eta}$$

Definition. Weingarten Operator

The expression

$$W_\eta(X) = -P(\partial_X \eta)$$

is called the shape or Weingarten operator.

It turns out that the Weingarten operator encodes the same information as the second fundamental form. Though it is a convenient form in certain cases, some books omit it because it does not give new information.
Proposition. Let X, Y be tangent vector fields and η a normal vector field of some surface. Then

$$\boxed{\langle W_\eta(X), Y\rangle = \langle B(X,Y), \eta\rangle}$$

Proof. Since $\eta \perp Y$, it follows that $\langle \eta, Y\rangle = 0$. But then $\partial_X \langle \eta, Y\rangle = 0$. Observe that

$$\begin{aligned}\langle \partial_X \eta, Y\rangle + \langle \eta, \partial_X Y\rangle &= 0\\ \langle P(\partial_X \eta) + (\mathrm{Id} - P)(\partial_X \eta), Y\rangle + \langle \eta, P(\partial_X Y) + (\mathrm{Id} - P)(\partial_X Y)\rangle &= 0\\ \langle P(\partial_X \eta), Y\rangle + \langle \eta, (\mathrm{Id} - P)(\partial_X Y)\rangle &= 0\\ -\langle W_\eta(X), Y\rangle + \langle \eta, B(X,Y)\rangle &= 0\end{aligned}$$

and our claimed equality follows.

□

Corollary. $W_\eta(X)$ is linear with respect to both arguments and $W_\eta(X)(A)$ depends only on η and X at A, that is, $W : N_A\Sigma \times T_A\Sigma \to T_A\Sigma$.
Proof. Recall the previous formula

$$\langle W_\eta(X), Y\rangle = \langle B(X,Y), \eta\rangle$$

The scalar product is linear in the second argument, η. Hence the expression on the left is linear with respect to η. Also, B is linear in both arguments and the scalar product is linear in the first argument. The composition of linear functions is linear, so this shows that the expression on the left is also linear in X.

□

If we fix a normal vector η, then $W_\eta : T_A\Sigma \to T_A\Sigma$ is a linear operator on a tangent space. This is why it is called an operator.

Definition. $\nabla^{N\Sigma}_X \eta$

The symbol $\nabla^{N\Sigma}_X \eta$ denotes the covariant derivative of a normal vector field η with respect to a tangent vector (or vector field) X. We have $\nabla : T_A\Sigma \times \Gamma(N\Sigma) \to N_A\Sigma$ or $\nabla : \Gamma(T\Sigma) \times \Gamma(N\Sigma) \to \Gamma(N\Sigma)$

Exercise. The following properties hold.

1. $$\nabla^{N\Sigma}_{f_1X_1+f_2X_2}\eta = f_1\nabla^{N\Sigma}_{X_1}\eta + f_2\nabla^{N\Sigma}_{X_2}\eta$$
2. $$\nabla^{N\Sigma}_X(\eta_1+\eta_2) = \nabla^{N\Sigma}_X\eta_1 + \nabla^{N\Sigma}_X\eta_2$$
3. $$\nabla^{N\Sigma}_X(f\eta) = \partial_X f \cdot \eta + f \cdot \nabla^{N\Sigma}_X\eta$$
4. There is no symmetricity because considering $\nabla_X\eta - \nabla_\eta X$, the second term is senseless. We cannot take a derivative along a normal vector, only tangent vectors.
5. $$\partial_X \langle \eta_1, \eta_2 \rangle = \langle \nabla_X\eta_1, \eta_2 \rangle + \langle \eta_1, \nabla_X\eta_2 \rangle$$

Definition. Weingarten Derivational Equation

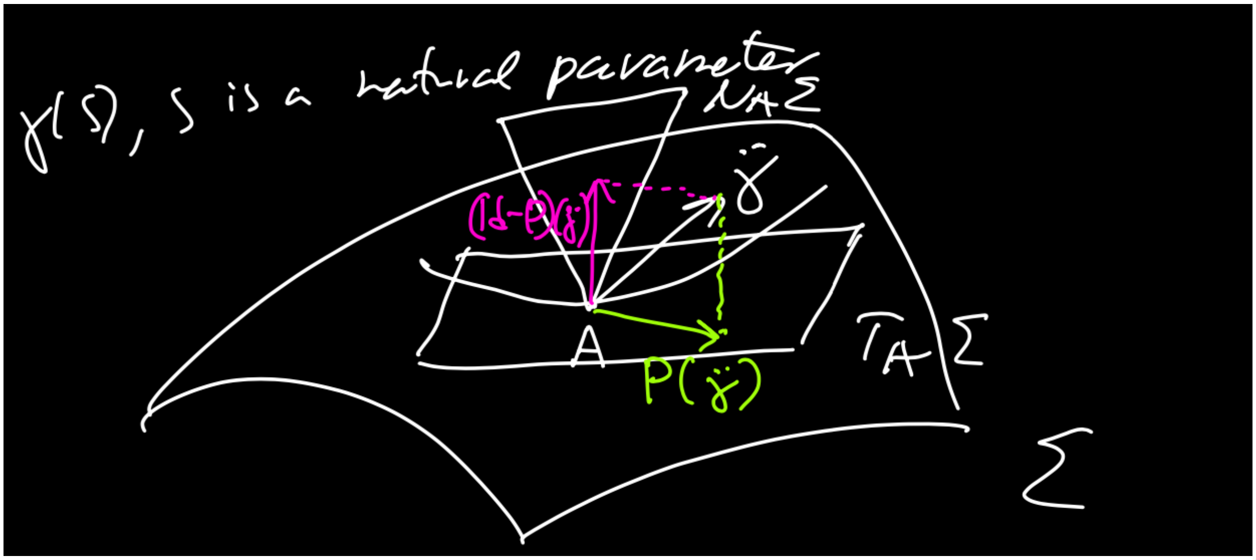
The equation is given by
$$\partial_X\eta = -W_\eta(X) + \nabla^{N\Sigma}_X\eta$$

Similarly, its particular case is the second Frenet equation
$$\partial_v n = -kv$$
where $W_n(v) = -kv$ and $\nabla^{N\Sigma}_v n = 0$.
The object $\nabla^{N\Sigma}$ is called a connection on the normal bundle.

3.4 Curves on k-dimensional surfaces in n-dimensional space

Consider the below setup again. Recall that $\ddot{\gamma} = \frac{d}{ds}\dot{\gamma} = \partial_{\dot{\gamma}}\dot{\gamma}$ from our previous proposition. As a result
$$k_n = |(\mathrm{Id}-P)(\ddot{\gamma})| = |(\mathrm{Id}-P)(\partial_{\dot{\gamma}}\dot{\gamma})| = |B(\dot{\gamma},\dot{\gamma})|$$
$$k_g = |P(\ddot{\gamma})| = |P(\partial_{\dot{\gamma}}\dot{\gamma})| = |\nabla_{\dot{\gamma}}\dot{\gamma}|$$

Proposition. If s is a natural parameter on a curve, then
$$k_n = |B(\dot{\gamma},\dot{\gamma})|, \quad k_g = |\nabla_{\dot{\gamma}}\dot{\gamma}|$$



An important point: objects in geometry may exist independently of coordinates. For example, considering a vector as an arrow or a linear operator as a map (and not its matrix representation). However, a precise description often requires coordinates.

How do we describe a covariant derivative?

Consider a basis $e_1, \ldots, e_k$ in the tangent vector fields. In this context, a basis refers to vector fields such that at any point A, the corresponding vectors $e_1(A), \ldots, e_k(A) \in T_A\Sigma$ are a basis in $T_A\Sigma$. An example is $r_{u^1}, \ldots, r_{u^k}$.
Let X, Y be tangent vector fields for a surface Σ. Then we may write $X = \sum_i X^i e_i$ and $Y = \sum_j Y^j e_j$. Observe that we switch from global coordinates of the vector field Y to local coordinates Y^j here.

Aside. Einstein Summation Convention.
If a formula contains an index twice, once as a subscript and once as a superscript, we assume a summation with respect to this index. As an example

$$X^i e_i = \sum_i X^i e_i \qquad X^i \frac{\partial r}{\partial x^i} = \sum_i X^i \frac{\partial r}{\partial x^i}$$

Now, consider the covariant derivative of X and Y, we see

$$\begin{aligned} \nabla_X Y &= \nabla_{X^i e_i} Y^j e_j = X^i \nabla_{e_i} (Y^j e_j) \\ &= X^i \left((\partial_{e_i} Y^j) e_j + Y^j \nabla_{e_i} e_j \right) \end{aligned}$$

We see that if we know $\nabla_{e_i} e_j$, then we can compute any $\nabla_X Y$. Notice that $\nabla_{e_i} e_j$ is a tangent vector field. Hence, it has coordinates in the basis e_i and we may write

$$\nabla_{e_i} e_j = \Gamma^p_{ij} e_p$$

We write the lower indices ij for Γ because it depends on e_i and e_j. Further the objects Γ^p_{ij} are called **Christoffel symbols**. We may think of this as a collection of smooth functions. Continuing our formula with the above substitution, we have

$$\nabla_X Y = X^i((\partial_{e_i} Y^j)e_j + Y^j \Gamma^p_{ij} e_p)$$

Notice that there is a triple summation for one of the terms. In the first summand, we may replace the index j with p, so $Y^j e_j$ becomes $Y^p e_p$. Then the formula becomes

$$\nabla_X Y = X^i(\partial_{e_i} Y^p + Y^j \Gamma^p_{ij})e_p$$

Proposition. We may express the covariant derivative using the formula

$$\nabla_X Y = X^i(\partial_{e_i} Y^p + Y^j \Gamma^p_{ij})e_p$$

or equivalently,

$$(\nabla_X Y)^p = X^i(\partial_{e_i} Y^p + Y^j \Gamma^p_{ij})$$

Moreover, for the basis, $r_{u^1}, \dots r_{u^k}$ we have

$$\nabla_X Y = X^i \left(\frac{\partial Y^p}{\partial u^i} + Y^j \Gamma^p_{ij} \right) r_{u^p}$$

recalling that $\partial_{r_{u^i}} = \frac{\partial}{\partial u^i}$ from our exercise class.
Thus, we see that Γ^p_{ij} works as matrix coefficients of ∇. The Christoffel symbols encode all the information about the covariant derivative.
Important: We can find Γ^p_{ij} by computing explicitly $\nabla_{e_i} e_j$ and then finding its coordinates, $\Gamma^p_{ij} = (\nabla_{e_i} e_j)^p$. But, in fact, we compute it in another way using the metric, i.e. the first fundamental form.

The quantity $\langle \nabla_X Y, Z \rangle$ is important because if this quantity is known for any vector field Z then we know $\nabla_X Y$. Consider an orthonormal basis $e_1, \dots, e_k$. Notice that $\langle \nabla_X Y, e_i \rangle$ is the i^{th} coordinate in this basis, and by our assumption, we know its value. Thus, we know every component of the vector $\nabla_X Y$ in this orthonormal basis.
Proposition. The following formula holds

$$(\nabla_X Y, Z) = \frac{1}{2}(\partial_X (Y, Z) + \partial_Y (Z, X) - \partial_Z (X, Y) + (Z, [X, Y]) + (Y, [Z, X]) - (X, [Y, Z]))$$

The above follows from $\partial_X (Y, Z) = (\nabla_X Y, Z) + (Y, \nabla_X Z)$ and $[X, Y] = \nabla_X Y - \nabla_Y X$, the properties of symmetry and metric compatibility.
Corollary. A metric (which in this context means the same as *scalar product* and *first fundamental form*) defines the covariant derivative, via the previous formula.

Consider a basis in tangent vector fields $e_1, \dots, e_k$. Denote the metric coefficients $g_{ij} = (e_i, e_j)$, these are elements of the matrix of the first fundamental form. Since the commutator of two tangent vector fields is again a tangent vector field, we may write

$$[e_i, e_j] = c_{ij}^p e_p$$

Now, take $X = e_i$, $Y = e_j$, and $Z = e_p$ in the previous formula

$$\begin{aligned}
(\nabla_{e_i} e_j, e_p) &= \frac{1}{2}(\partial_{e_i}(e_j, e_p) + \partial_{e_j}(e_p, e_i) - \partial_{e_p}(e_i, e_j) + (e_p, [e_i, e_j]) + (e_j, [e_p, e_i]) - (e_i, [e_j, e_p])) \\
(\Gamma_{ij}^q e_q, e_p) &= \frac{1}{2}(\partial_{e_i} g_{jp} + \partial_{e_j} g_{pi} - \partial_{e_p} g_{ij} + (e_p, c_{ij}^q e_q) + (e_j, c_{pi}^q e_q) - (e_i, c_{jp}^q e_q)) \\
\Gamma_{ij}^q g_{qp} &= \frac{1}{2}(\partial_{e_i} g_{jp} + \partial_{e_j} g_{pi} - \partial_{e_p} g_{ij} + c_{ij}^q g_{pq} + c_{pi}^q g_{jq} - c_{jp}^q g_{iq})
\end{aligned}$$

Aside. We often write the elements of the Gram matrix of $e_1, \dots, e_k$ as $G = (g_{ij})$ and its inverse $G^{-1} = (g^{ij})$, where the position of the indices indicates whether or not the matrix is an inverse matrix. Also, we have $GG^{-1} = I \iff g_{ij} g^{jp} = \delta_i^p$ and in the same way $G^{-1}G = I \iff g^{ij} g_{jp} = \delta_p^i$

So, multiplying both sides by the inverse matrix, we have

$$\begin{aligned}
\Gamma_{ij}^m &= \Gamma_{ij}^q \delta_q^m = \Gamma_{ij}^q g_{qp} g^{pm} \\
&= \frac{1}{2} g^{pm}(\partial_{e_i} g_{jp} + \partial_{e_j} g_{pi} - \partial_{e_p} g_{ij} + c_{ij}^q g_{pq} + c_{pi}^q g_{jq} - c_{jp}^q g_{iq})
\end{aligned}$$

Remark. Recall that the definition of the Christoffel symbol is $\Gamma_{ij}^p e_p = \nabla_{e_i} e_j = P(\partial_{e_i} e_j)$. In practice, it is difficult to compute the projector when the basis is not orthonormal, so the formula we derived is much more convenient.

There are particular cases of the above formula for the Christoffel symbol which are important.

1. $e_1, \dots, e_k$ is orthonormal. Let us introduce the notation $c_{ij,p} = c_{ij}^q g_{pq}$. Since the basis is orthonormal, we have $g_{ij} = \delta_{ij}$. Since δ_{ij} is always a constant, it follows that $\partial_{e_l} g_{ij} = 0$. As a result, the first three terms vanish and we have

$$\Gamma_{ij}^m = \frac{1}{2} g^{pm}(c_{ij,p} + c_{pi,q} - c_{jp,i})$$

2. We have the basis $e_1 = \mathbf{r}_{u^1}, \dots, e_k = \mathbf{r}_{u^k}$. For this basis, we have shown in exercise class that $[e_i, e_j] = 0$. This means that $c_{ij}^p = 0$. Also, we know that $\partial_{e_i} f = \frac{\partial f}{\partial u^i}$. Putting it

together, our formula simplifies to

$$\begin{aligned}\Gamma_{ij}^m &= \frac{1}{2}g^{pm}(\partial_{e_i}g_{jp} + \partial_{e_j}g_{pi} - \partial_{e_p}g_{ij}) \\ &= \frac{1}{2}g^{pm}\left(\frac{\partial g_{jp}}{\partial u^i} + \frac{\partial g_{pi}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^p}\right) \\ &= \frac{1}{2}g^{pm}\left(\frac{\partial g_{pj}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^p} + \frac{\partial g_{ip}}{\partial u^j}\right)\end{aligned}$$

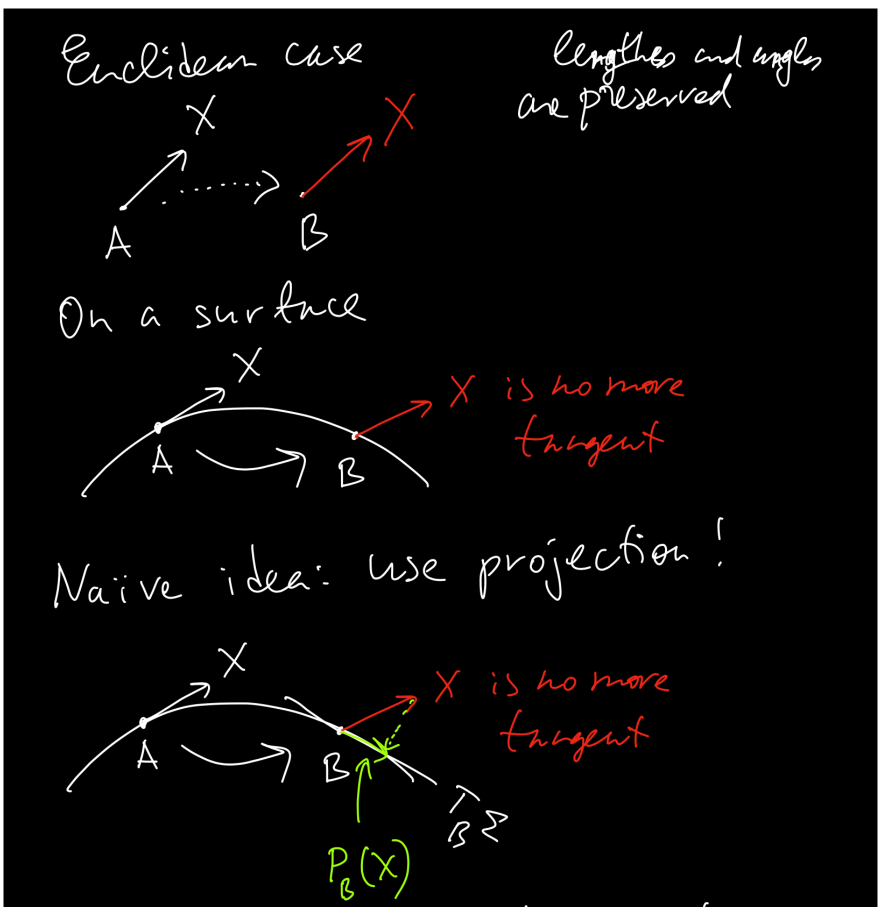
Note that there is a summation in this expression over p. A memory device involves switching the p in the middle term with each index i, j separately.

3.5 Applications of the Covariant Derivative

Two common applications of the covariant derivative are parallel transport and geodesics. We begin with parallel transport. The length and angle (relative to the x-axis) are preserved.

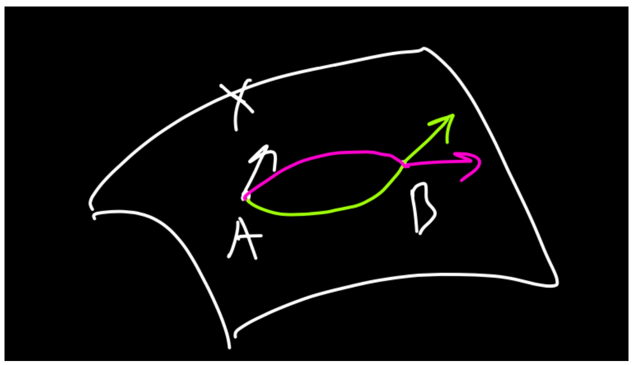
Parallel Transport

The idea of parallel transport in the Euclidean case is trivial, we simply add by the vector with which we want to translate.



On a surface, if we attempt the same translation to a tangent vector, then tangency is no longer retained by this operation. To be precise, the naïve approach of taking $X \mapsto P_B(X)$, where $X \in T_A\Sigma$ and $P_B(X) \in T_B\Sigma$ does not preserve lengths and angles. This is important because in later settings, we study differential geometry on abstract manifolds. Abstract manifolds are not embedded in an ambient space and there are only tangent vectors and no normal vectors. In order for our operation of parallel transport to be generalizable, we need tangency to be preserved.

In the Euclidean case, we see that parallel transport preserves lengths and angles, but on a surface, it is not possible to preserve lengths, angles, and tangency without sacrificing something (this impossibility is related to curvature). The thing we sacrifice is that instead of parallel transport being from two points A and B, it will now be from two points A and B along a curve on the surface.



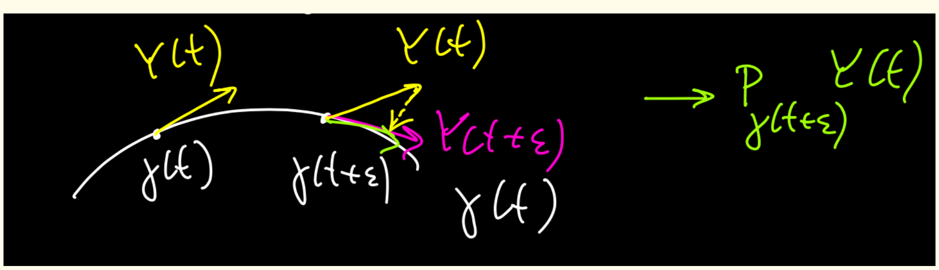
Now, let us consider a surface Σ and a vector field $Y(t) \in T_{\gamma(t)}\Sigma$, the vector field tangent to Σ, but defined only on a curve $\gamma(t)$. Note that $T_{\gamma(t)}\Sigma$ is a tangent space, it is distinct from the velocity vector field of γ.

Definition. Parallel

A vector field $Y(t)$ is **parallel** along a curve $\gamma(t)$ if for all t

$$Y(t+\varepsilon) = P_{\gamma(t+\varepsilon)}Y(t) + O(\varepsilon)$$

for $\varepsilon \to 0$.



We note that there are many functions that are equal up to an $O(\varepsilon)$ term, such as $\sin x$ and x near 0. However, the requirement that the function satisfies this at every point t is a more restrictive one. We now derive a convenient equivalence to a vector field being parallel. Since $Y(t+\varepsilon)$ is already tangent, we know that $P_{\gamma(t+\varepsilon)}(Y(t+\varepsilon)) = Y(t+\varepsilon)$. Then, we may rewrite the condition for a parallel vector field as

$$P_{\gamma(t+\varepsilon)}(Y(t+\varepsilon) - Y(t)) = O(\varepsilon)$$

Recall from Taylor approximation, we have

$$f(t+\varepsilon) = f(t) + \varepsilon f'(t) + O(\varepsilon)$$

Substituting for f, we have

$$\begin{aligned} P_{\gamma(t+\varepsilon)} &= P_{\gamma(t)} + \varepsilon \frac{d}{dt}(P_\gamma(t)) + O(\varepsilon) \\ Y(t+\varepsilon) - Y(t) &= \varepsilon \frac{dY}{dt} + O(\varepsilon) \end{aligned}$$

Making substitutions into our original expression, we have

$$\begin{aligned} P_{\gamma(t+\varepsilon)}(Y(t+\varepsilon) - Y(t)) &= \left(P_{\gamma(t)} + \varepsilon \frac{d}{dt}(P_\gamma(t)) + O(\varepsilon)\right)\left(\varepsilon \frac{dY}{dt} + O(\varepsilon)\right) \\ &= \varepsilon P_{\gamma(t)}\left(\frac{dY}{dt}\right) \\ &= O(\varepsilon) \end{aligned}$$

where we have used $\varepsilon^2 \cdot f = O(\varepsilon)$ and $O(\varepsilon) + O(\varepsilon) = O(\varepsilon)$ to make the simplifications. Dividing the last equality above through by ε, we see that

$$P_{\gamma(t)}\left(\frac{dY}{dt}\right) = O(1)$$

Taking the limit as $\varepsilon \to 0$ gives $P_{\gamma(t)}\left(\frac{dY}{dt}\right) = 0$. Now, since $\frac{dY}{dt} = \partial_{\dot\gamma} Y$, we observe that

$$\nabla_{\dot\gamma} Y = P(\partial_{\dot\gamma} Y) = 0$$

Proposition. A vector field $Y(t)$ is parallel along the curve $\gamma(t)$ if and only if $\nabla_{\dot\gamma} Y = 0$.
When we take the derivative with respect to Y, it must be defined not only on the points of the curve γ, but also in some neighborhood. Since the result $\nabla_{\dot\gamma} Y = 0$ is equivalent to $P_{\gamma(t)}\left(\frac{dY}{dt}\right) = 0$, and the latter does not depend on the extension of Y in the neighborhood, the former expression also does not depend on the extension.

Definition. Parallel Transport

The object $\nabla_{\dot\gamma} Y = 0$ is called the equation of parallel transport (transfer).

Recall that we discussed the formula

$$\nabla_X Y = X^i(\partial_{e_i} Y^j + Y^p \Gamma^j_{ip})\, e_j$$

Also, for the basis $e_i = r_{u^i}$, we have

$$\partial_{e_i} = \frac{\partial}{\partial u^i}$$

Hence, we may write

$$\nabla_X Y = X^i \left(\frac{\partial Y^j}{\partial u^i} + Y^p \Gamma^j_{ip} \right) r_{u^j}$$

There is also a corresponding coordinate representation. In coordinates, the curve has a form like $\gamma(t) = (u^1(t), \ldots, u^k(t))$. Then taking the derivative of the curve, we have

$$\dot{r} = r_{u^1}\dot{u}^1 + \ldots, + r_{u^k}\dot{u}^k$$

and this is equivalent to the fact that $\dot{\gamma}(t)$ has coordinates $(\dot{u}^1, \ldots, \dot{u}^k)$ in the basis $r_{u^1}, \ldots, r_{u^k}$. Hence, we have the following equivalences

$$\nabla_{\dot{\gamma}} Y = 0 \iff \forall j \left((\nabla_{\dot{\gamma}} Y)^j = 0 \right) \iff \forall j \left(\dot{u}^i \left(\frac{\partial Y^j}{\partial u^i} + Y^p \Gamma^j_{ip} \right) = 0 \right)$$

Distributing $\dot{u}^i$, we see that for all j

$$\frac{\partial Y^j}{\partial u^i} \frac{du^i}{dt} + Y^p \Gamma^j_{ip} \dot{u}^i = 0$$

Finally, via the chain rule, we have for all j

$$\frac{dY^j}{dt} + Y^p \Gamma^j_{ip} \dot{u}^i = 0$$

This is the equation of parallel transport written in coordinates.

Remark. Consider what we know in the above equation. The Christoffel symbols are known because they are defined by a point on our surface. The $\dot{u}^i$ are known because our curve is given. The parallel field Y is not known. From these unknowns, we see that we have a system of linear first order ordinary differential equations. With all the arguments explicit, we have

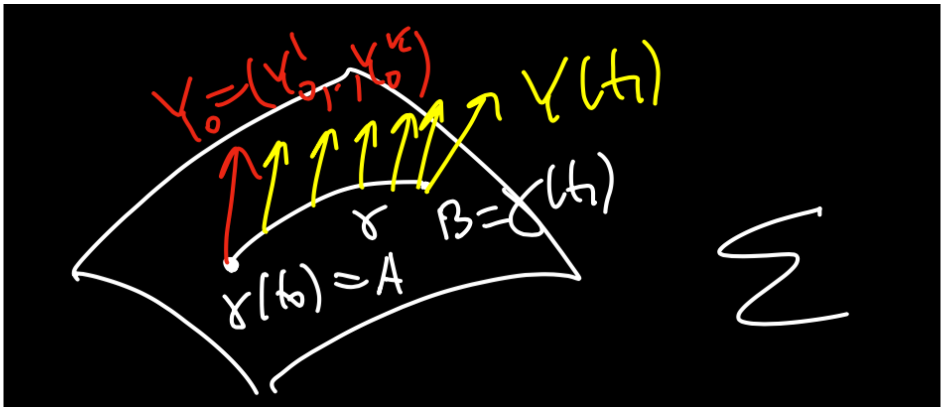
$$\frac{dY^j}{dt}(t) + Y^p(t)\, \Gamma^j_{ip}(u^1(t), \ldots, u^k(t))\, \dot{u}^i(t) = 0$$

We have a Cauchy problem with the following system and initial conditions

$$\begin{cases} \dot{Y}^j + Y^p \Gamma^j_{ip} \dot{u}^i = 0 \\ Y^i(t_0) = Y^i_0 \end{cases}$$

Theorem. Let Σ be a surface and $\gamma(t)$ be a curve on Σ. Let $Y_0 \in T_{\gamma(t_0)}\Sigma$. Then there exists a unique parallel vector field $Y(t)$ along $\gamma(t)$ such that $Y(t_0) = Y_0$.

This theorem is a corollary of the fact that we have a system of linear ODEs and that the Cauchy problem has a unique global solution in this case. The situation is as in the picture below. For some initial data, there exists a unique parallel vector field along the curve γ.



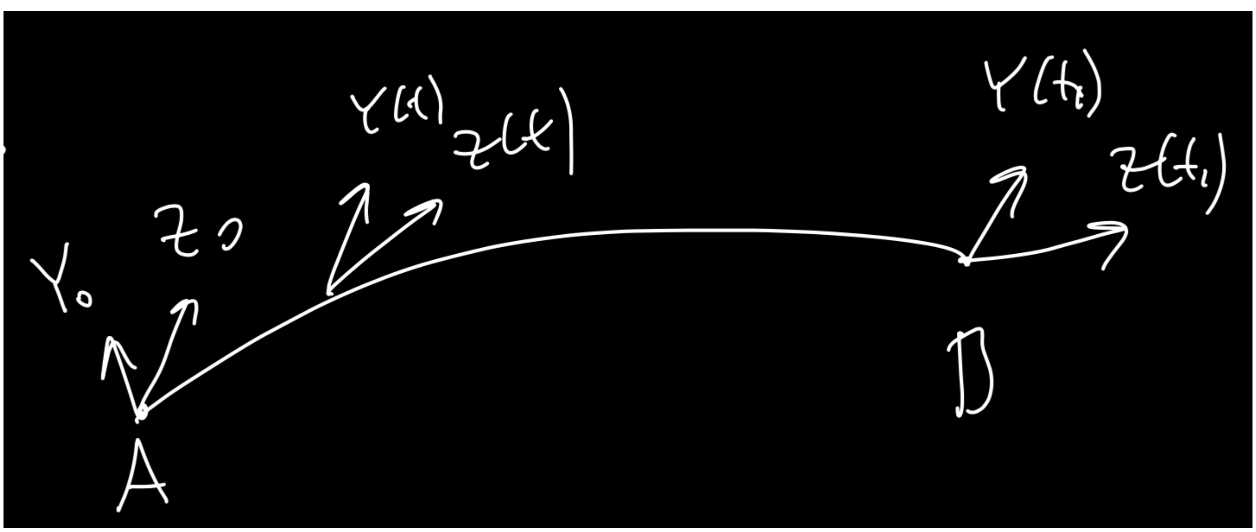
Definition. Result of a Parallel Transport

The vector $Y(t_1) \in T_{\gamma(t_1)}\Sigma$ is called the result of a parallel transport of $Y_0 \in T_{\gamma(t_0)}\Sigma$ from the point $\gamma(t_0)$ to $\gamma(t_1)$ along $\gamma(t)$

By definition, we know that the vector will be in the tangent plane of the curve after translation. How do we know that lengths and angles are preserved?

Theorem. Parallel transport preserves lengths and angles.

Proof. Consider the following diagram.



We want to prove that

$$(Y(t_1), Z(t_1)) = (Y_0, Z_0)$$

Taking the derivative, we have

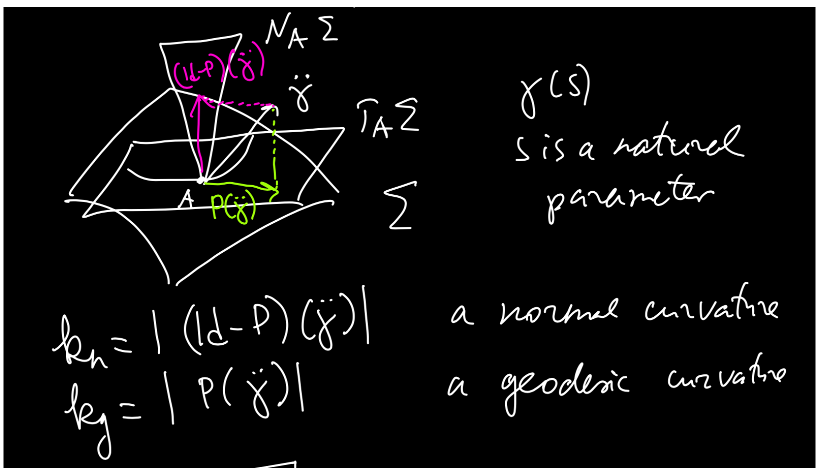
$$\frac{d}{dt}(Y(t), Z(t)) = \partial_{\dot{\gamma}}(Y, Z)$$
$$= (\nabla_{\dot{\gamma}} Y, Z) + (Y, \nabla_{\dot{\gamma}} Z) = 0$$

This shows that the scalar product $(Y(t), Z(t))$ is a constant. Thus, its value is the same at both t_0 and t_1.

■

Geodesics

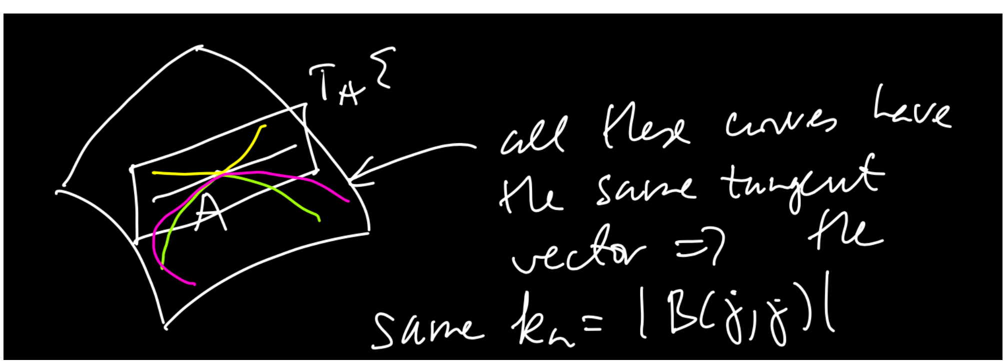
The intuitive idea of geodesics is that they are straight lines on curved surfaces, we may also approach it as the shortest distance between two points on a surface. Recall the following setup:



Also, by the Pythagorean theorem we have $k = \sqrt{k_n^2 + k_g^2}$. Since $\partial_{\dot{\gamma}}\dot{\gamma} = \ddot{\gamma}$, it also follows that

$$k_n = |B(\dot{\gamma}, \dot{\gamma})| \qquad k_g = |\nabla_{\dot{\gamma}}\dot{\gamma}|$$

Recall that we proved previously that a curve has zero curvature if and only if it is a straight line. Since there are no straight lines on a sphere, this means that there are no curves of curvature 0 on a sphere. We might ask what is the minimum curvature on a surface?



Consider the following diagram above. Since the normal curvature k_n depends only on the tangent vector at a point, all the curves through point A have the same normal curvature. Hence, among these curves, the minimal curvature occurs for a curve $k_g = 0$. Our reasoning is that $k = \sqrt{k_g^2 + k_n^2}$ and that k_n is the same for all curves passing through A, but the geodesic curvature k_g may vary.

Definition. Geodesic

A curve with $k_g = 0$ is called a geodesic curve (or simply a geodesic).

Suppose γ is a geodesic curve. Since $k_g = 0$ and $k_g = |\nabla_{\dot{\gamma}}\dot{\gamma}|$, it follows that $|\nabla_{\dot{\gamma}}\dot{\gamma}| = 0$. A vector has 0 length if and only if it is the zero vector. Hence $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$.

Definition. Equation of Geodesics

$\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ is called the equation of geodesics.

Recall that this equation is very similar to the equation for parallel transport, $\nabla_{\dot{\gamma}} Y = 0$. Recall that in local coordinates, it is

$$\dot{Y}^p + \Gamma_{ij}^p Y^j \dot{u}^i = 0$$

Thus, to obtain a similar equation for the geodesic in the style of the parallel transport equation in local coordinates, we should take $Y = \dot{\gamma}$. Letting $\gamma(t) = (u^1(t), \ldots, u^k(t))$, we obtain

$$\ddot{u}^p + \Gamma_{ij}^p \dot{u}^i \dot{u}^j = 0$$

Aside. Concerning notation, what is happening is we are considering the curve

$$(r \circ \gamma)(t) = r(u^1(t), \ldots, u^k(t))$$

When we take the derivative of this curve with respect to t, we have

$$\frac{d}{dt}(r \circ \gamma) = r_{u^i} \dot{u}^i$$

In local coordinates, the coefficients are $\dot{u}^1, \ldots, \dot{u}^k$. So $\dot{Y}^p = \ddot{u}^p$ based on this.

Proposition. The equation of geodesics in local coordinates is

$$\ddot{u}^p(t) + \Gamma_{ij}^p(u^1(t), \ldots, u^k(t)) \cdot \dot{u}^i(t)\dot{u}^j(t) = 0$$

While this equation seems similar to the equation of parallel transport, the key difference is that only the vector field Y is unknown in parallel transport. For geodesics, since the surface is known, the Christoffel symbols Γ_{ij}^p are known. However, a curve $(u^1(t), \ldots, u^k(t))$

is unknown.

Observation. The equation of geodesics is a system of second order **nonlinear** ODEs.

Cauchy Problem for the Equation of Geodesics

The problem is specified by the system of equations with initial data

$$\begin{aligned} \ddot{u}^p + \Gamma^p_{ij}\dot{u}^i\dot{u}^j &= 0 \\ u^p(t_0) &= A^p \\ \dot{u}^p(t_0) &= \mathrm{X}^p \end{aligned}$$

where $p \in \{1, \dots, k\}$ and $\dim \Sigma = k$. Geometrically, on our surface Σ, we have a point A which our curve passes through with velocity vector X at the point A.

> **Theorem.** Existence of Geodesic Curve.
> Given a surface Σ, a point $A \in \Sigma$, a tangent vector $\mathrm{X} \in T_A\Sigma$ and some $t_0 \in \mathbb{R}$, there exists a unique curve $\gamma : (a, b) \to \Sigma$ with $t_0 \in (a, b)$ such that γ is a geodesic, $\gamma(t_0) = A$, and $\dot{\gamma}(t_0) = \mathrm{X}$.

There is a detail concerning the word *unique* when applied to curves. Given any solution, we can restrict the domain, which is an interval, to a smaller subset and this restriction is technically distinct from the original solution. However, the sense of unique here means the largest interval where this solution is defined.

Proposition. $\dot{\gamma}$ is parallel along a geodesic γ.

Proof. If γ is geodesic, then

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0$$

which also means that $\dot{\gamma}$ is parallel. Recall the proposition.

■

Corollary. The length of the velocity vector of a geodesic γ is constant, that is

$$|\dot{\gamma}| = c$$

for some constant c.

Proof. Using the above proposition $\dot{\gamma}$ is a parallel vector field, and parallel transport preserves lengths. ■

Observation. Since $|\dot{\gamma}(t)| = c$, consider another parameter $s = ct$. Then, by the chain rule

$$\left|\frac{d\gamma}{ds}\right| = \left|\frac{d\gamma}{dt}\frac{dt}{ds}\right| = 1$$

Hence, $s = ct$ is a natural parameter. This shows us that t is a natural parameter up to some rescaling.

Definition. Affine Natural Parameter

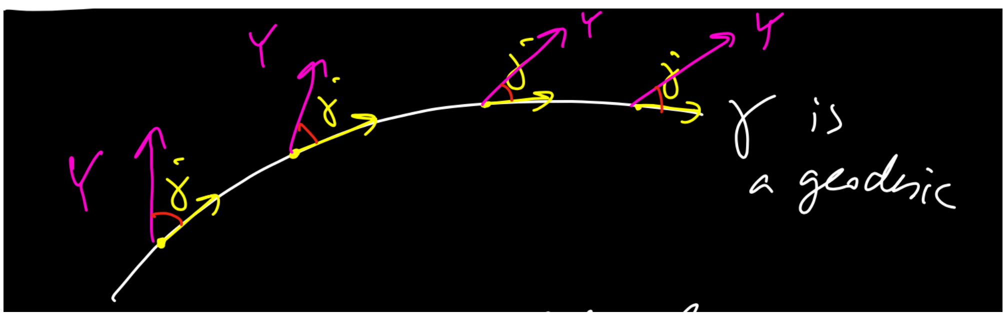
Let s be a natural parameter. Then a parameter $t = as + b$, where $a > 0$, is called an affine natural parameter. It is called as such because it is the formula for the affine transformation of a real line.

Proposition.

1. If $\gamma(t)$ is a solution of the equation of geodesics, then t is an affine natural parameter on the geodesic. Note that being a solution to the equation of geodesics imposes a restriction on the parameter t.

2. $\langle \dot{\gamma}, \dot{\gamma} \rangle = |\dot{\gamma}|^2 = c$, the scalar product of $\dot{\gamma}$ is a constant. This object is called the **first integral** of the equation of geodesics.

From the theory of ODEs, if there are enough first integrals, then it is possible to solve the differential equation through integration, hence their name.



Proposition. If Y is parallel along a geodesic γ, then the angle between Y and $\dot{\gamma}$ is preserved.

Proof. Since γ is geodesic, $\dot{\gamma}$ is parallel. Parallel transport preserves lengths and angles. ∎

Proposition. γ is a geodesic if and only if $\ddot{\gamma} \perp T_{\gamma(t)}\Sigma$ or $\ddot{\gamma} = 0$ for all t.

Proof. Notice that

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0 \iff P(\partial_{\dot{\gamma}}\dot{\gamma}) = 0 \iff P(\ddot{\gamma}) = 0$$

Since the projection of $\ddot{\gamma}$ to the tangent space has no contribution, it means that $\ddot{\gamma} \perp T_{\gamma(t)}\Sigma$ or $\ddot{\gamma} = 0$.

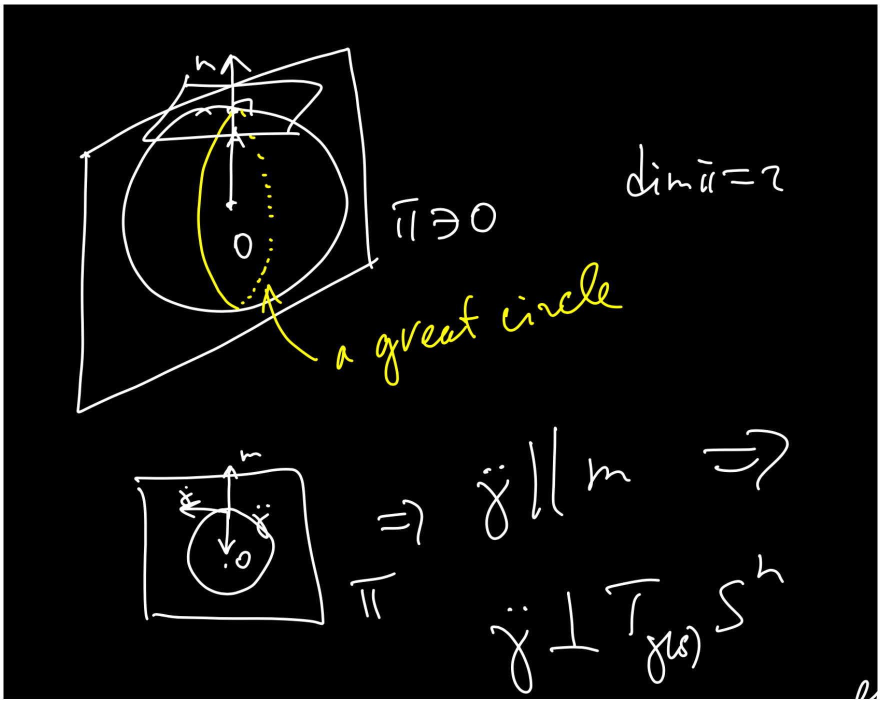
∎

Corollary. If a straight line lies on a surface, then it is a geodesic. We should take an affine natural parameter in this case.

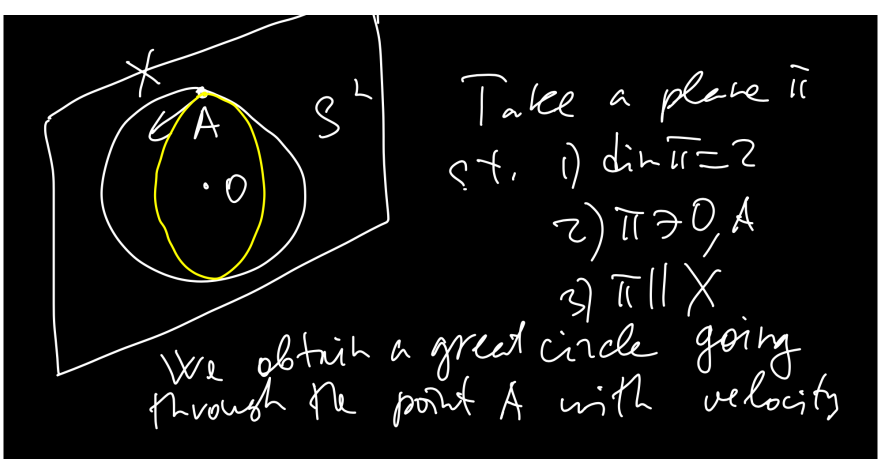
Proof. If γ is a straight line, then $\ddot{\gamma} = 0$. By the previous proposition, γ is a geodesic. ∎

Examples Involving Geodesics

A simple example of the above corollary are straight lines on the surface of a cylinder.
Now, consider a sphere $S^n \subseteq \mathbb{R}^{n+1}$. Recall that the intersection of a two-dimensional plane Π with the sphere containing the center of the circle is called a great circle. In the picture below, we see that the acceleration vector $\ddot{\gamma}$ points is parallel to the position vector of the point on the sphere.



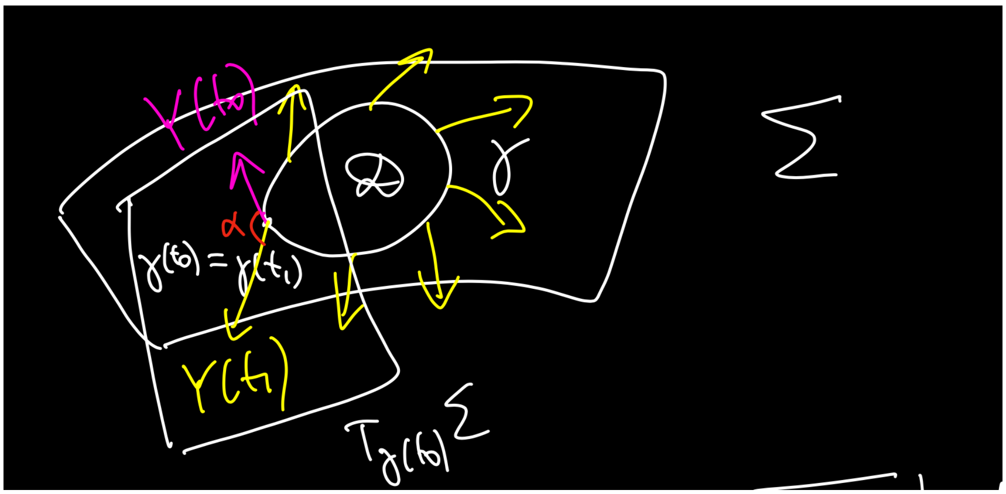
Since the position vector and the normal vector to the tangent plane at a point are also parallel (the normal vector is just a continuation of the position vector), it follows that $\ddot{\gamma} \perp T_{\gamma(s)}S^n$. Hence, a great circle (with an affine natural parameter) is a geodesic.
Consider a similar situation where we take a 2-dimensional plane Π through the sphere S^n. The points O, A in the plane and the velocity vector X is parallel to the plane.



With this setup, we obtain a great circle going through the point A with velocity vector X. But from the previous argument, we know that a great circle is a geodesic. By the uniqueness property, this great circle is the only geodesic corresponding to this point A and tangent vector X. However, since we may make this argument for any point A on the sphere and any tangent vector X at that point, it follows that all geodesics on the sphere must be great circles. The moral of the story is to think about geometry before attempting to solve ODEs, since the particular system of ODEs for geodesics is a second order nonlinear ODE. Often, we must resort to guessing what the geodesic (i.e. the curve) is, and then confirming that it satisfies the system.

Preview of Upcoming Topics

We have the parallel transport of $Y(t_0)$ along the closed curve γ in the diagram below.



Suppose that $\gamma(t_0) = \gamma(t_1)$. Then $Y(t_0)$ and $Y(t_1)$ must be at the same point and since Y is a tangent vector field along γ, these two vectors must belong to the same tangent plane.

Theorem. Letting α be the angle between $Y(t_0)$ and $Y(t_1)$, the following equality holds:

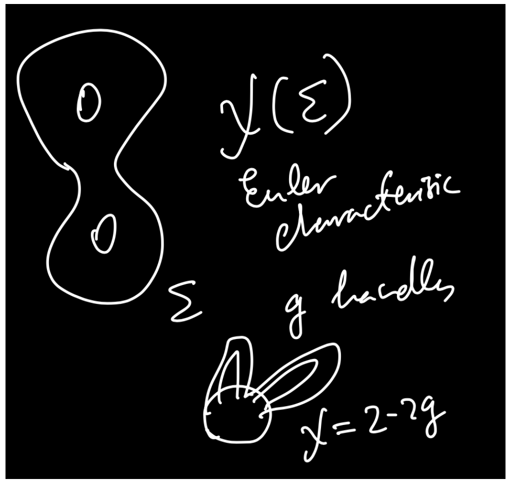
$$\alpha = \iint_{\mathscr{D}} K\, dS$$

where

$$dS = \sqrt{EG - F^2}\, du \wedge dv$$

and $\mathscr{D}$ is the region bounded by the closed curve and K is Gaussian curvature.

The above theorem is not only interesting in itself, but it has many corollaries.
Suppose we have a closed surface without a boundary. There is a purely topological property of the surface called its Euler characteristic, denoted $\chi(\Sigma)$.



All closed surfaces without a boundary are homeomorphic to a sphere with g handles, and have $\chi(\Sigma) = 2 - 2g$.

Theorem. (Gauss-Bonnet). It turns out that the Euler characteristic for a closed surface without a boundary may be computed in the following way

$$\chi(\Sigma) = \frac{1}{2\pi} \iint_{\Sigma} K \, dS$$

This theorem is remarkable because it reveals a bridge between a purely topological concept with an analytical one.

3.6 More on the Derivational Equations

We now turn to a more in-depth discussion of the Gauss and Weingarten derivational equations, which will be necessary for the two theorems stated immediately prior.
Let X, Y be tangent vector fields and η be a normal vector field. We recall the Gauss and Weingarten derivational formulas, respectively:

$$\partial_X Y = \nabla_X Y + B(X, Y)$$
$$\partial_X \eta = -W_\eta(X) + \nabla_X^{N\Sigma} \eta$$

Take some $e_1, \ldots, e_k$, a basis in tangent vector fields. Currently, we know that we may express the covariant derivative of these basis vector fields in terms of a linear combination involving Christoffel symbols Γ_{ij}^p

$$\nabla_{e_i} e_j = \Gamma_{ij}^p e_p$$

Put another way, the Christoffel symbols completely describe the covariant derivative. Using the properties of the Levi-Civita connection, we have already deduced that

$$\nabla_X Y = X^i \left(\partial_{e_i} Y^p + \Gamma_{ij}^p Y^j \right) e_p$$

How do we describe the second fundamental form, B? We may represent X and Y in the basis tangent vector fields as $X = X^i e_i$ and $Y = Y^j e_j$. Then, using the bilinearity of B, we have

$$B(X, Y) = B(X^i e_i, Y^j e_j) = X^i Y^j B(e_i, e_j)$$

Thus, if we know $B(e_i, e_j)$, then we will be able to compute $B(X, Y)$. We know that $B(e_i, e_j)$ is a normal vector, so to describe it we need a basis in the normal vector field. Let $\eta_1, \dots, \eta_{n-k}$ be a basis in normal vector fields, i.e. a set of normal vector fields such that for any point $A \in \Sigma$, the vectors $\eta_1(A), \dots, \eta_{n-k}(A)$ form a basis of the corresponding normal space $N_A\Sigma$. Then, in an analogous fashion to Christoffel symbols, we may write

$$B(e_i, e_j) = b_{ij}^{\nu} \eta_\nu$$

We will use Greek letters to refer to the normal objects (i.e. vectors, spaces, indices) and Roman letters to refer to tangent objects. Note that i, j are enumerating tangent vectors, so they range from 1 to k, where $k = \dim \Sigma$. In contrast, $\nu = 1, \dots, n-k$, since ν enumerates the normal basis vectors, where $n - k = n - \dim \Sigma = \dim N_A\Sigma$. So now we may write

$$B(X, Y) = B(X^i e_i, Y^j e_j) = X^i Y^j b_{ij}^{\nu} \eta_\nu$$

Definition. Coefficients of the Second Fundamental Form

The coefficients b_{ij}^{ν} are called the **coefficients of the second fundamental form**. They are defined by the formula below.

$$B(X, Y) = X^i Y^j b_{ij}^{\nu} \eta_\nu$$

We can then write the Gauss derivational formula for basis vectors

$$\begin{aligned} \partial_{e_i} e_j &= \nabla_{e_i} e_j + B(e_i, e_j) \\ &= \Gamma_{ij}^{p} e_p + b_{ij}^{\nu} \eta_\nu \end{aligned}$$

Particular cases.

1. $k = 1$, $n = 2$. This is the case of planar curves. Then we have $e_1 = v$ and $\eta_1 = n$, from our theory of planar curves earlier in the semester. Our equations are

$$\partial_v v = \Gamma_{11}^1 v + b_{11}^1 n = kn$$

In this case, we have $\Gamma_{11}^1 = 0$ and $b_{11}^1 = k$.

2. $k = 2$, $n = 3$. This is the case of 2-dimensional surfaces in $\mathbb{R}^3$. Suppose that our surface is given by the parametrization $r(u, v)$. Then $e_1 = r_u$, $e_2 = r_v$ and $\eta_1 = m$. Recall that $B(X, Y) = \mathrm{II}(X, Y)\, m$. Observe that

$$B(e_1, e_1) = \mathrm{II}(r_u, r_u)\, m = Lm$$

Thus, it follows that $b^1_{11} = L = (r_{uu}, m)$. In the same way

$$b^1_{12} = b^1_{21} = M = (r_{uv}, m) = (r_{vu}, m)$$
$$b^1_{22} = N = (r_{vv}, m)$$

Recall that $\partial_{r_u} f = \frac{\partial f}{\partial u}$ and $\partial_{r_u} r_u = r_{uu}$. Now, we write

$$\partial_{e_1} e_1 = \Gamma^p_{11} e_p + b^1_{11}\eta_1$$
$$r_{uu} = \partial_{r_u} r_u = \Gamma^1_{11} r_u + \Gamma^2_{11} r_v + Lm$$

and in the same way

$$r_{uv} = \Gamma^1_{12} r_u + \Gamma^2_{12} r_v + Mm$$
$$r_{uv} = r_{vu} = \Gamma^1_{21} r_u + \Gamma^2_{21} r_v + Mm$$
$$r_{vv} = \Gamma^1_{22} r_u + \Gamma^2_{22} r_v + Nm$$

Why is this set of equations interesting? We already know that we can determine the Christoffel symbols using only the first fundamental form. We can directly write down a representation of these derivatives and focus on computing their coefficients (this is easier if the basis is orthonormal).

Remark. Recall that in Euclidean geometry, there are three properties relating to lines.

1. There exists a line between any two points.
2. The line is unique.
3. The line segment between the two points is the shortest path between these points.

These three properties fail for geodesics when we consider them globally relative to the surface the geodesic lies on. However, they hold if we consider the geodesic locally.

In the previous lecture, we have discussed the Gauss derivational equation for basis vectors

$$\boxed{\partial_{e_i} e_j = \Gamma^p_{ij} e_p + b^\nu_{ij}\eta_\nu}$$

We continue with a discussion of the Weingarten derivational equation.

$$\partial_X \xi = -W_\xi(X) + \nabla^{N\Sigma}_X \xi$$

Observe that $\nabla^{N\Sigma}$ behaves in a similar way to ∇. Given normal and tangent basis vector fields such that

$$X = \sum_{i=1}^{k} X^i e_i \qquad \xi = \sum_{\alpha=1}^{n-k} \xi^\alpha \eta_\alpha$$

By linearity and the product rule, we may obtain

$$\begin{aligned}\nabla^{N\Sigma}_X \xi &= \nabla^{N\Sigma}_{X^i e_i}(\xi^\alpha \eta_\alpha) = X^i \nabla^{N\Sigma}_{e_i}(\xi^\alpha \eta_\alpha) \\ &= X^i(\partial_{e_i} \underbrace{\xi^\alpha \eta_\alpha}_{\alpha \to \beta} + \xi^\alpha \underbrace{\nabla^{N\Sigma}_{e_i} \eta_\alpha}_{\kappa^\beta_{i\alpha}\eta_\beta})\end{aligned}$$

We propose a similar construction to Christoffel symbols, but for the covariant derivative of the normal vector field.

$$\nabla^{N\Sigma}_{e_i} \eta_\alpha = \kappa^\beta_{i\alpha} \eta_\beta$$

The symbols κ^β_{ia} are called the **local coefficients of** $\nabla^{N\Sigma}$. Our expression becomes

$$\nabla^{N\Sigma}_X \xi = X^i(\partial_{e_i} \xi^\beta + \xi^\alpha \kappa^\beta_{i\alpha})\eta_\beta$$

which is analogous to our earlier derivation. What happens if we want to compute the Weingarten derivational formula for basis vectors?

$$\partial_{e_i} \eta_\alpha = -W_{\eta_\alpha}(e_i) + \nabla^{N\Sigma}_{e_i} \eta_\alpha = -W_{\eta_\alpha}(e_i) + \kappa^\beta_{i\alpha} \eta_\beta$$

Recalling that $\langle W_\xi(X), Y \rangle = \langle \xi, B(X, Y) \rangle$, we deduce

$$\begin{aligned}\langle W_\xi(e_i), e_j \rangle &= \langle \xi, B(e_i, e_j) \rangle \\ &= \left\langle \xi^\alpha \eta_\alpha, b^\beta_{ij} \eta_\beta \right\rangle \\ &= \xi^\alpha b^\beta_{ij} \underbrace{\langle \eta_\alpha, \eta_\beta \rangle}_{g_{\alpha\beta}} = \xi^\alpha b^\beta_{ij} g_{\alpha\beta}\end{aligned}$$

where $g_{\alpha\beta}$ indicates an entry in the Gram matrix of $\eta_1, \ldots, \eta_{n-k}$.
Let us denote $W_{\eta_\alpha}(e_i) = W^p_{\alpha i} e_p$. We can express $W_{\eta_\alpha}(e_i)$ in terms of tangent basis vectors because it is the projection onto the tangent plane, $-P(\partial_{e_i} \eta_\alpha)$. Then

$$\xi^\alpha W^p_{\alpha i} \underbrace{\langle e_p, e_j \rangle}_{g_{pj}} = \xi^\alpha b^\beta_{ij} g_{\alpha\beta}$$

$$\xi^\alpha W^p_{\alpha i} g_{pj} = \xi^\alpha b^\beta_{ij} g_{\alpha\beta}$$

It is important to realize that g_{pj}, an entry in the Gram matrix of the system of tangent basis vector $e_1, \ldots, e_k$, is a **distinct** symbol from $g_{\alpha\beta}$. We can distinguish based on whether the indices are Greek or Latin letters. Recall also the convention that g^{ab} indicates an element of the inverse matrix, g^{-1}. Thus, $GG^{-1} = I \iff g_{pj} g^{jb} = \delta^b_p$. We obtain

$$W^p_{\alpha i} \delta^b_p = W^p_{\alpha i} g_{pj} g^{jb} = b^\beta_{ij} g_{\alpha\beta} g^{jb}$$

We finally arrive at our desired formula:

$$\boxed{W^p_{\alpha i} = b^\beta_{ij} g_{\alpha\beta} g^{jb}}$$

The point of deriving this formula was to be able to express our Weingarten derivational equation in normal basis vector fields in a different way, as below

$$\boxed{\begin{aligned}\partial_{e_i}\eta_\alpha &= -W_{\eta_\alpha}(e_i) + \nabla^{N\Sigma}_{e_i}\eta_\alpha \\ &= -b^{\beta}_{ij} g_{\alpha\beta} g^{jp} e_p + \kappa^{\beta}_{i\alpha}\eta_\beta\end{aligned}}$$

Note that we have written $g^{jp} = g^{jb}$ to make the Einstein summation make sense.

> *Aside.* Q: How do we know that there is a suitable matrix to multiply with G^{-1} on the right hand side? Matrix multiplication is not the only type of operation which may be captured through the Einstein summation convention. The point is that the terms corresponding some index have equal values. So, when we multiply each term by coefficients that are equal, the overall equality remains.

4 Differential Forms

4.1 Linear Algebra Preface and Differential Forms

Let V be a vector space. An element $v \in V$ is called a vector. Let V^* be the dual space. An element $\xi \in V^*$ is called a covector. A covector is a linear functional on V, so $\xi : V \to \mathbb{R}$. Given $v \in V$ and $\xi \in V^*$, we see that $\xi(v) \in \mathbb{R}$. The dual space is a vector space because addition of linear functionals and multiplication of linear functionals by a scalar are consistent with the vector space axioms.

In a finite-dimensional situation, there are many simplifications. For instance, if $\dim V = n$, then $\dim V^* = n$. Further, it is true that $(V^*)^* = V$ for finite-dimensional V as well. A vector may be paired with a covector, as below:

$$\xi(v) = \langle \xi, v \rangle$$

The idea is that this pairing is symmetric: ξ is a function which acts on vectors, but we may also think of applying v to ξ. This means thinking of v as a linear functional on covectors.

Aside. Let V be a finite-dimensional vector space. The rigorous idea of applying v to ξ is the **evaluation functional**, given by

$$\hat{v}(\xi) = \xi(v)$$

We may verify $\hat{v}$ is indeed linear, hence, $\hat{v} \in V^{**}$. Further, we may define another function $\psi : V \to V^{**}$ by

$$\psi(v) = \hat{v}$$

It turns out that ψ is an isomorphism.

Let $e_1, \dots, e_n$ be a basis in V. So, if $v \in V$, then $v = v^i e_i$.

Definition. Dual Basis

A basis $e^1, \dots, e^n$ of V^*, note the upper indices, is called a **dual basis** to $e_1, \dots, e_n$ if

$$e^i(e_j) = \delta^i_j$$

The dual basis consists of the set of **coordinate functionals** relative to $e_1, \dots, e_n$.

Proposition. For each basis $e_1, \dots, e_n$ of V, there exists a *unique* dual basis $e^1, \dots, e^n$ of V^*.

Definition. Polylinear Function

We call τ a **polylinear function** on V of k arguments if

$$\tau : \underbrace{V \times \cdots \times V}_{k} \to \mathbb{R}$$

such that it is linear with respect to each argument. To be explicit, this means

$$\begin{aligned}
\tau(\alpha_1 v_1 + \alpha_1' v_1', v_2, \ldots, v_n) &= \alpha_1 \tau(v_1, v_2, \ldots, v_n) + \alpha_1' \tau(v_1', v_2, \ldots, v_n) \\
\tau(v_1, \alpha_2 v_2 + \alpha_2' v_2', v_3, \ldots, v_n) &= \ldots \\
&\vdots
\end{aligned}$$

This is sometimes also called a **multilinear function**.

There are two important classes of polylinear functions: symmetric and antisymmetric.
Examples.

1. A bilinear form (symmetric)
2. A scalar product (symmetric)
3. The determinant of a matrix A, is polylinear with respect to the rows or columns of the matrix. It is antisymmetric.

Proposition. The set $T_k V$ of polylinear functions with k arguments on V is a linear (vector) space.

Proof. The addition of two polylinear functions and multiplication of polylinear functions by a scalar defined in the usual (pointwise) way produce again polylinear functions.

□

Remark. The symbol $S^2 V^* \subseteq T_2 V$ denotes the linear subspace of *symmetric* bilinear forms. Also, $T_2 V$ is the vector space of bilinear forms.

Definition. Skew-Symmetric

The function $\tau \in T_k V$ is skew-symmetric (or antisymmetric) if swapping any two arguments changes the sign:

$$\tau(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\tau(v_1 \ldots, v_j, \ldots, v_i, \ldots, v_k)$$

Examples.

1. The determinant

2. Oriented area in $\mathbb{R}^2$, since

$$\text{OrArea}(a,b) = \det \begin{vmatrix} a^1 & b^1 \\ a^2 & b^2 \end{vmatrix}$$

We denote the subspace of skew-symmetric polylinear functions of k arguments as $\wedge^k V^* \subseteq T_k V$.

Definition. Wedge (Exterior) Product

The wedge product takes a skew-symmetric form of k arguments and l arguments and produces a skew-symmetric form of $k+l$ arguments.

$$\wedge : \wedge^k V^* \times \wedge^l V^* \to \wedge^{k+l} V^*$$

Consider $\sigma \in \wedge^k V^*$ and $\tau \in \wedge^l V^*$. Then $\sigma \wedge \tau \in \wedge^{k+l} V^*$. What is the result of the wedge product? There are two conventions.

$$(\sigma \wedge \tau)(v_1, \ldots, v_{k+l}) = \frac{1}{(k+l)!} \sum_{\pi \in S_{k+l}} \text{sgn}(\pi) \cdot \sigma(v_{\pi(1)}, \ldots, v_{\pi(k)}) \cdot \tau(v_{\pi(k+1)}, \ldots, v_{\pi(k+l)})$$

The alternative convention uses the coefficient $\frac{1}{k!l!}$ instead.

Example. Let $k = l = 1$. So $S_{1+1} = S_2 = \{(1,2),(2,1)\}$. The permutation group contains the identity permutation and the permutation which exchanges the elements. The sign is $+1, -1$ for these permutations, respectively. By our definition

$$(\sigma \wedge \tau)(v_1, v_2) = \frac{1}{2}(\sigma(v_1)\tau(v_2) - \sigma(v_2)\tau(v_1))$$

Notice that this expression is skew-symmetric. If we exchange v_1 and v_2, we get by substitution $-(\sigma \wedge \tau)(v_1, v_2)$. So, we have verified that given $\sigma, \tau \in V^* = \wedge^1 V^*$ and $\sigma \wedge \tau \in \wedge^2 V^*$.

Definition. Degree

If $\sigma \in \wedge^k V^*$, we say that that the degree of σ is k, that is

$$\deg \sigma = k$$

Properties of the Wedge Product.

1.
$$(\sigma_1 + \sigma_2) \wedge \tau = \sigma_1 \wedge \tau + \sigma_2 \wedge \tau$$
$$\lambda(\sigma \wedge \tau) = \lambda\sigma \wedge \tau = \sigma \wedge \lambda\tau$$
$$\sigma \wedge (\tau_1 + \tau_2) = \sigma \wedge \tau_1 + \sigma \wedge \tau_2$$

These properties follow immediately from the definition. It is bilinear.

2.
$$\sigma \wedge \tau = (-1)^{\deg \sigma \deg \tau} \tau \wedge \sigma$$

For each individual term, we want to consider $\operatorname{sgn}(\pi)\, \sigma \cdot \tau = \operatorname{sgn}(\pi')\, \tau \cdot \sigma$. However, then we must compose π with another permutation $(k+1 \ \ldots \ k+l \ 1 \ 2 \ \ldots k)$ to achieve the ordering of arguments when we switch σ and τ. But then

$$\operatorname{sgn}(\pi') = \operatorname{sgn}(k+1 \ \ldots \ k+l \ 1 \ 2 \ \ldots k) \cdot \operatorname{sgn}(\pi) = (-1)^{\deg \sigma \deg \tau} \operatorname{sgn}(\pi)$$

Here, $\deg \sigma \deg \tau = kl$ is the number of transpositions needed to move $k+1$ to the first position, $k+2$ to the second position, and so forth.

3.
$$(\sigma \wedge \tau) \wedge \rho = \sigma \wedge (\tau \wedge \rho)$$

Associativity follows from some difficult algebraic and combinatorial reasoning.

Remark. A real number may be considered a function of zero arguments, $\mathbb{R} = \wedge^0 V^*$. It follows that for $\lambda \in \mathbb{R}$ and $\sigma \in \wedge^k V^*$, we have

$$\lambda \wedge \sigma = \frac{1}{k!} \sum_{\pi \in S_k} \operatorname{sgn}(\pi) \cdot \lambda \cdot \sigma(v_{\pi(1)}, \ldots, v_{\pi(k)}) = \lambda \sigma$$

Recall that $\operatorname{sgn}(\pi)$ is defined as (-1) raised to the parity of the number transpositions which compose the permutation π, and σ is antisymmetric. Hence, $\operatorname{sgn}(\pi) \cdot \sigma(v_{\pi(1)}, \ldots, v_{\pi(k)}) = \sigma(v_1, \ldots, v_k)$.

Proposition. If $e_1, \ldots, e_n$ is a basis in V, then $e^{i_1} \wedge \ldots \wedge e^{i_k}$ where $i_1 < \ldots < i_k$ forms a basis in $\wedge^k V^*$. This is why the notation for antisymmetric polylinear functions involves the dual space, V^*.

Corollary. If $\sigma \in \wedge^k V^*$, then it may be expressed as a linear combination of the basis elements of $\wedge^k V^*$, that is

$$\sigma = \sum_{i_1 < \ldots < i_k} \sigma_{i_1 \ldots i_k} \cdot e^{i_1} \wedge \ldots \wedge e^{i_k}$$

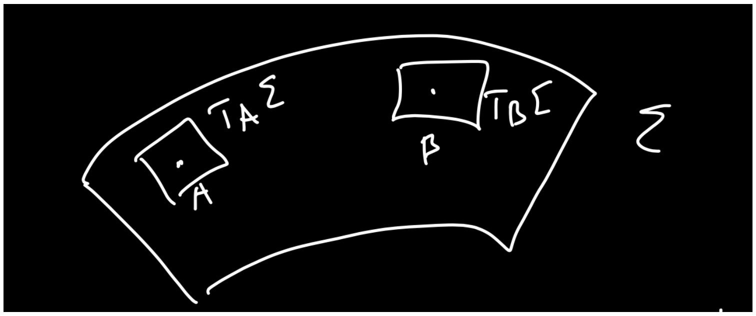
Example. If we consider $\wedge^2 \mathbb{R}^3$, then the basis is $e^1 \wedge e^2, e^1 \wedge e^3, e^2 \wedge e^3$. This may be expressed as

$$e^{i_1} \wedge \cdots \wedge e^{i_k}$$

This means that the basis for $\wedge^k V^*$ is of size $\binom{n}{k}$.

4.2 Application to Surfaces and Manifolds

Instead of dealing with skew-symmetric polylinear functions in one vector space, we now deal with them along each of the tangent planes to the points on a surface.



We call $T_A\Sigma$ the tangent space to Σ at a point A and we denote its dual space $T_A^*\Sigma := (T_A\Sigma)^*$. We call this dual space the cotangent space to Σ at a point A.

Previously, we used the idea of tangent vector fields to capture having a tangent vector at each point on the surface. Let X be a tangent vector field. This means that for every point $A \in \Sigma$, we have $X(A) \in T_A\Sigma$.

We may consider an analogous situation where we consider a covector field, ω. This means that for every $A \in \Sigma$, $\omega_A \in T_A^*\Sigma$. To be explicit, at a point A on Σ, there is a linear function ω_A defined on the tangent plane $T_A\Sigma$ and at another point B on Σ, there is another linear function ω_B defined on the tangent plane $T_B\Sigma$. So, a covector field specifies a linear function on each tangent space. A covector field is often called a differential 1-form.

Remark. If X is a vector field and ω a differential 1-form, then $\omega(X) : \Sigma \to \mathbb{R}$ is a function satisfying

$$\omega(X)(A) = \omega_A(X_A) \in \mathbb{R}$$

where $\omega_A \in T_A^*\Sigma$ and $X_A \in T_A\Sigma$. It appears that this is simply linear algebra at different points up until now, but there are new operations on surfaces.

Notation. We call $\Omega^1(\Sigma)$ the space of 1-forms on Σ.

Definition. Differential

A differential $d : C^\infty(\Sigma) \to \Omega^1(\Sigma)$ of a function f is a differential 1-form df such that

$$df(X) = \partial_X f$$

Proposition. Let $u^1, \ldots, u^k$ be local coordinates on a surface Σ. Take $e_1 = r_{u^1}, \ldots, e_k = r_{u^k}$ as a basis in the tangent vector fields. Then $du^1, \ldots, du^k$ form the dual basis in the differential 1-forms.

Proof. Observe that

$$du^i(r_{u^j}) = \partial_{r_{u^j}} u^i = \frac{\partial u^i}{\partial u^j} = \delta^i_j$$

So we have shown that du^i is the set of coordinate functionals relative to $r_{u^1}, \ldots, r_{u^k}$, satisfying the definition of dual basis. Note that $du^i(X) = X^i$ while $u^i(A) = A^i$. □

Corollary. If $\omega \in \Omega^1(\Sigma)$, then

$$\omega = \omega_1 du^1 + \ldots + \omega_k du^k = \omega_i du^i$$

Proposition. We may write the differential as

$$df = \frac{\partial f}{\partial u^1} du^1 + \ldots + \frac{\partial f}{\partial u^k} du^k = \frac{\partial f}{\partial u^i} du^i$$

Proof. Since $df \in \Omega^1(\Sigma)$, it follows that $df = w_i du^i$ for some coefficients w_i. How do we determine these coefficients? Notice that

$$w_j = w_i \delta^i_j = w_i \cdot du^i(r_{u^j}) = df(r_{u^j}) = \partial_{r_{u^j}} f = \frac{\partial f}{\partial u^j}$$

□

Remark. This is the formula we have learned in multivariable calculus. One difference is that in calculus, this was a formula only in a flat (Euclidean) space, rather than on a surface. However, now the meaning is clear: we have a 1-form which is represented using a linear combination in the basis for the space of 1-forms, where du^i are the basis vectors and $\frac{\partial f}{\partial u^i}$ are the coefficients. It has nothing to do with infinitesimals. At each point in the corresponding cotangent space, it is just an expression of the covector in the basis.

Definition. k-form

A field of skew-symmetric polylinear functions of k arguments on $T_A\Sigma$ is called a differential k-form. Again, the degree of the form refers to the number of arguments. This space is denoted as $\Omega^k(\Sigma)$

So, we know that $\wedge : \Omega^k(\Sigma) \times \Omega^l(\Sigma) \to \Omega^{k+1}(\Sigma)$ and that $du^{i_1} \wedge \ldots \wedge du^{i_k}$ is a basis in $\Omega^k(\Sigma)$. Hence, for $\omega \in \Omega^k(\Sigma)$, we may write

$$w = \sum_{i_1 < \ldots < i_k} w_{i_1 \cdots i_k} \cdot du^{i_1} \wedge \ldots \wedge du^{i_k}$$

Example. For $\omega \in \Omega^2(\mathbb{R}^3)$, we may write

$$\omega = \omega_{12}\, du^1 \wedge du^2 + \omega_{13}\, du^1 \wedge du^3 + \omega_{23}\, du^2 \wedge du^3$$

Traditionally, this is notated as

$$P\, dy \wedge dz + Q\, dz \wedge dx + R\, dx \wedge dy$$

But, these are the same thing.

Remark. In linear algebra, we have $\mathbb{R} = \wedge^0 V^*$, $V^* = \wedge^1 V^*$, and for two arguments $\wedge^2 V^*$. What happens on a surface? We have smooth functions which are the 0-forms, $C^\infty(\Sigma) = \Omega^0(\Sigma)$, and after that we have 1-forms $\Omega^1(\Sigma)$, 2-forms $\Omega^2(\Sigma)$, and so on. Is it possible to define the differential in a way that each time we apply it to a k-form, the result is a $(k+1)$-form?

$$\Omega^0(\Sigma) \xrightarrow{d} \Omega^1(\Sigma) \xrightarrow[?]{d} \Omega^2(\Sigma) \xrightarrow[?]{d} \Omega^3(\Sigma) \to \cdots$$

The answer is yes, which the next theorem demonstrates.

Theorem. There exists a unique operator

$$d : \Omega^k(\Sigma) \to \Omega^{k+1}(\Sigma)$$

such that

1. For smooth functions f, we have $df(X) = \partial_X f$
2. $d(\sigma + \tau) = d\sigma + d\tau$
3. $d(\sigma \wedge \tau) = d\sigma \wedge \tau + (-1)^{\deg \sigma} \sigma \wedge d\tau$
4. $d^2\sigma = 0$

Proof. The proof is shown in the exercise class. □

4.3 Pullback and Push Forward

The following statements apply not only to surfaces but more also generally to smooth manifolds. Let M, N be two surfaces. Suppose we have a map $F : M \to N$, where $u^1, \dots, u^m$ are local coordinates in M and $v^1, \dots, v^n$ are local coordinates for N. We may think of F as being given by

$$\begin{aligned} v^1 &= v^1(u^1, \dots, u^m) \\ &\vdots \\ v^n &= v^n(u^1, \dots, u^m) \end{aligned}$$

When we have a map of two surfaces (or manifolds), then all objects are travelling in some direction, that is from M to N or vice versa. Objects consist of

1. **points**, $p \mapsto F(p)$
2. **functions**, consider

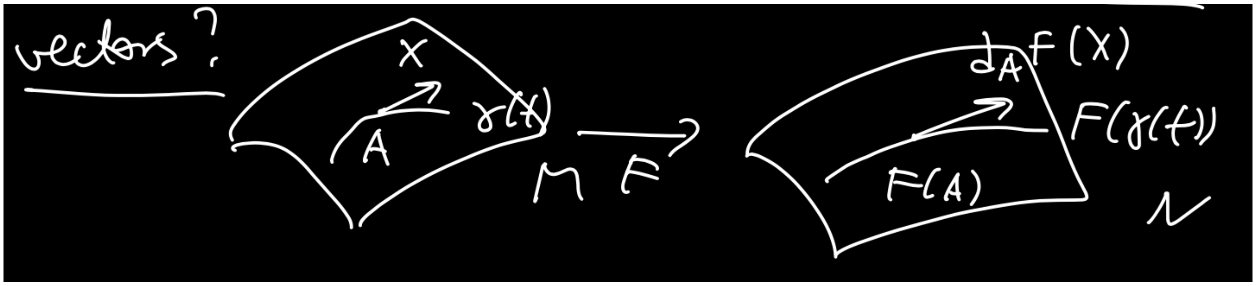
   $$M \overset{F}{\mapsto} N \overset{\varphi}{\mapsto} \mathbb{R}$$

   for some $\varphi \in \mathscr{C}^\infty(N)$. There is also the composition $F^*\varphi = \varphi \circ F \in \mathscr{C}^\infty(M)$. We have an induced map of functions

   $$\mathscr{C}^\infty(M) \overset{F^*}{\hookleftarrow} \mathscr{C}^\infty(N)$$

   The function $F^*\varphi$ is called a pullback of a function $\varphi \in \mathscr{C}^\infty(N)$ by a map F.

3. **vectors**. We consider velocity vectors of curves on the surface. Vectors travel in the same direction as points, from one tangent space to another.



We call $\gamma(0) = A$ as our point on the surface and $\dot{\gamma}(0) = X$ as the tangent vector to the point. Since points are mapped between surfaces by F, the curve $\gamma(t)$ is also mapped into the surface N as the image of these points $F(\gamma(t))$. Since $F(\gamma(0)) = F(A)$, it follows that

$$\left.\frac{d}{dt}\right|_{t=0} F(\gamma(t)) \in T_{F(A)}N$$

Definition. $d_A F(X)$

This is called the differential of a map F at a point A. The function maps between tangent spaces as below

$$T_A M \xmapsto{d_A F} T_{F(A)}N$$

We define

$$d_A F(X) := \left.\frac{d}{dt}\right|_{t=0} F(\gamma(t))$$

Note that this is well-defined: it does not depend on the curve γ.

Exercise. We show that $d_A F(X)$ is well-defined. If $\gamma_1(0) = \gamma_2(0) = A$ and $\dot{\gamma}_1(0) = \dot{\gamma}_2(0) = X$, then

$$\left.\frac{d}{dt}\right|_{t=0} F(\gamma_1(t)) = \left.\frac{d}{dt}\right|_{t=0} F(\gamma_2(t))$$

Exercise. On M we have chosen local coordinates $u^1, \ldots, u^m$. Thus, there is a basis $r^M_{u^1}, \ldots, r^M_{u^m} \in T_A M$. Then, $X \in T_A M$ has coordinates $(X^1, \ldots, X^m)$ in this basis. Similarly, there is a basis $r^N_{v^1}, \ldots, r^N_{v^n} \in T_{F(A)}N$ associated with the local coordinates $v^1, \ldots, v^n$. Also, $d_A F(X) \in T_{F(A)}N$ has coordinates $(d_A F(X)^1, \ldots, d_A F(X)^n)$ in this basis. Now, we may write

$$\begin{pmatrix} d_A F(X)^1 \\ \vdots \\ d_A F(X)^n \end{pmatrix} = \begin{pmatrix} \frac{\partial v^1}{\partial u^1}(A) & \ldots & \frac{\partial v^1}{\partial u^m}(A) \\ \vdots & \ddots & \vdots \\ \frac{\partial v^n}{\partial u^1}(A) & \ldots & \frac{\partial v^n}{\partial u^m}(A) \end{pmatrix} \begin{pmatrix} X^1 \\ \vdots \\ X^m \end{pmatrix}$$

Corollary. $d_A F$ is a linear map $T_A M \to T_{F(A)}M$ with a matrix equal to the Jacobian matrix of a map F.

From last lecture, we discussed the differential of a function, now we have discussed a differential of a map, how are they related?
Suppose we have a function $f : M \to \mathbb{R}$ and we choose t as coordinate on $\mathbb{R}$, then

$$d_A f : T_A M \to T_{f(A)}\mathbb{R} \simeq \mathbb{R}$$

So, since $T_{f(A)}\mathbb{R}$ is isomorphic to $\mathbb{R}$, we may consider $d_A F$ as a covector, since it is a map from a vector space to the scalars. Also, recall that the formula is

$$d_A f = \frac{\partial f}{\partial u^i} du^i$$

In fact, this shows that the differential of a function is simply a particular case of the differential of a map, they are the same thing.
Let us now consider what happens to the object of *differential forms*. Recall that smooth functions, elements of $\mathscr{C}^\infty(M) = \Omega^0(M)$ or $\mathscr{C}^\infty(N) = \Omega^0(N)$ in our case, are 0-forms because the do not depend on the tangent vector. We have stated earlier that functions travel in the opposite direction as points via pullbacks, but in the language of forms, we have

$$\Omega^0(M) \xleftarrow{F^*} \Omega^0(N)$$

This is true for forms of any degree:

$$\Omega^k(M) \xleftarrow{F^*} \Omega^k(N)$$

Consider that $\omega \in \Omega^k(N) \mapsto F^*\omega \in \Omega^k(M)$. The expression $F^*\omega$ is called the pullback of ω by a map F. If $X_1, \ldots, X_k \in T_A M$, then

$$(F^*\omega)_A(X_1, \ldots, X_k) = \omega_{F(A)}(d_A F(X_1), \ldots, d_A F(X_k))$$

This is a very important and common notion in mathematics. In linear algebra, we have seen something similar in the form of the *conjugate operator*. Let $V \xrightarrow{A} W$ be a linear map and $V^* \xleftarrow{A^*} W^*$ be the conjugate map of A. Let $A^*\xi \in V^*$ and $\xi \in W^*$. Notice that

$$(A^*\xi)(v) = \xi(Av)$$
$$\langle A^*\xi, v\rangle = \langle \xi, Av\rangle$$

Exercise. Properties of F^*.

1. $F^*(\sigma_1 + \sigma_2) = F^*\sigma_1 + F^*\sigma_2$
2. $F^*(\sigma \wedge \tau) = F^*\sigma \wedge F^*\tau$
3. $F^*(d\omega) = d(F^*\omega)$

Definition. Differential Algebra

Let

$$\Omega(M) = \bigoplus_{k=0} \Omega^k(M)$$

Then, $\Omega(M)$ is a vector space with addition on forms and multiplication via $\lambda \in \mathbb{R}$. When we add on forms, we are considering it as a formal sum where only forms with the same degree are added together - it is not possible to add forms of different degrees. Next, it is a ring via the wedge product $\wedge$. With both these structures together, we have an algebra of differential forms on M. With the operation $d : \Omega(M) \to \Omega(M)$ added to the previous structure, we have what is called a differential algebra.

Observe that the pullback F^* preserves all the operations of the differential algebra. Thus, we may say that $F^* : \Omega(N) \to \Omega(M)$ is a homomorphism of differential algebras.

Corollary. A practical way to find $F^*\omega$ is to just substitute. We explain the meaning of this with the following example.

Example. Consider $F : \mathbb{R}^2 \to \mathbb{R}^2$ where we map from x, y coordinates to u, v coordinates. Suppose that F is given by

$$\begin{cases} u = x^2 + y^2 \\ v = xy \end{cases}$$

Consider some $\omega \in \Omega^2(N)$ given by

$$\omega = (u^2 + v)\, du \wedge dv$$

How do we compute $F^*\omega$?

$$\begin{aligned} F^*\omega &= F^*((u^2 + v)\, du \wedge dv) = F^*(u^2 + v)F^*(du) \wedge F^*(dv) \\ &= F^*(u^2 + v)dF^*u \wedge dF^*v \end{aligned}$$

Here, we have reduced everything to the pullback of functions. We then use the definition $F^*\omega = \omega \circ F$ and make a substitution to see

$$\begin{aligned} F^*(u^2 + v)\, dF^*u \wedge dF^*v &= ((x^2 + y^2)^2 + xy)\, d(x^2 + y^2) \wedge d(xy) \\ &= (x^4 + 2x^2y^2 + y^4 + xy)(2xdx + 2ydy) \wedge (ydx + xdy) \\ &= (x^4 + 2x^2y^2 + y^4 + xy)(2x^2 - 2y^2)\, dx \wedge dy \end{aligned}$$

One very important particular case of a pullback is change of coordinates.

$$\underset{u^1,\dots,u^m}{M} \xrightarrow{\text{Id}} \underset{v^1,\dots,v^m}{M}$$

Here, we are mapping one set of local coordinates to another. The identity transformation means that points are mapped to the same point, but they are represented in different coordinates now. With functions, we will obtain how these functions are expressed with a new set of coordinates.

Example. Consider $\underset{r,\varphi}{\mathbb{R}^2} \xrightarrow{\text{Id}} \underset{x,y}{\mathbb{R}^2}$. We have the usual mapping $x = r\cos\varphi$ and $y = r\sin\varphi$. Now, consider the form

$$\omega = dx \wedge dy$$

How do we write this in polar coordinates? We simply consider

$$\text{Id}^*\omega = \omega$$

We have

$$\begin{aligned}
dx \wedge dy &= d(r\cos\varphi) \wedge d(r\sin\varphi) \\
&= (\cos\varphi\, dr - r\sin\varphi d\varphi) \wedge (\sin\varphi\, dr + r\cos\varphi d\varphi) \\
&= r\cos^2\varphi dr \wedge d\varphi - r\sin^2\varphi d\varphi \wedge dr \\
&= r dr \wedge d\varphi
\end{aligned}$$

which is the familiar factor we get when we switch to polar coordinates during integration.

4.4 Integration of Differential Forms

The Riemann integral is an important operation and it may be generalized in many different ways, one of these ways which is germane to differential geometry is that we are integrating differential forms of degree equal to the dimension of the space.

Let us recall the formula of change of variables in an integral from calculus

$$\int_\Omega f(u_1, \ldots, u_n)\, du_1 \ldots du_n = \int_{\Omega'} f(v_1, \ldots, v_n) \cdot |\det J|\, dv_1 \ldots dv_n$$

Suppose we have M, a surface, with local coordinates $u^1, \ldots, u^n$. Since the dimension of the space is 1, we know that any form is proportional to

$$f(u^1 \ldots, u^n)\, du^1 \wedge \ldots \wedge du^n$$

Now, we perform a change of variables:

$$\begin{aligned}
u^1 &= u^1(v^1, \ldots, v^n) \\
&\vdots \\
u^n &= u^n(v^1, \ldots, v^n)
\end{aligned}$$

So, we substitute to get

$$\begin{aligned}
&f(u^1(v^1,\ldots,v^n)\ldots,u^n(v^1,\ldots,v^n))\cdot du^1(v^1,\ldots,v^n)\wedge\ldots\wedge du^n(v^1,\ldots,v^n)\\
&= f(\ldots)\left(\frac{\partial u^1}{\partial v^1}dv^1+\ldots+\frac{\partial u^1}{\partial v^n}dv^n\right)\wedge\ldots\wedge\left(\frac{\partial u^n}{\partial v^1}dv^1+\ldots+\frac{\partial u^n}{\partial v^n}dv^n\right)\\
&= f(\ldots)\left(\sum_{\pi\in S_n}\frac{\partial u^1}{\partial v^{\pi(1)}}dv^{\pi(1)}\wedge\ldots\wedge\frac{\partial u^n}{\partial v^{\pi(n)}}dv^{\pi(n)}\right)\\
&= f(\ldots)\left(\sum_{\pi\in S_n}\operatorname{sgn}(\pi)\frac{\partial u^1}{\partial v^{\pi(1)}}\cdots\frac{\partial u^n}{\partial v^{\pi(n)}}\right)dv^1\wedge\ldots\wedge dv^n\\
&= f(\ldots)\cdot\det J\cdot dv^1\wedge\ldots\wedge dv^n
\end{aligned}$$

where the equality when we distribute sums over the wedge products follows since any wedge product of linearly dependent vectors is 0. So, the only way to obtain something nonzero is to have some permutation of dv^i.

Aside. It is a famous equality that the complete expansion of the determinant of an $n\times n$ matrix A with entries A_{ij} is given by

$$\det A=\sum_{\pi\in S_n}\operatorname{sgn}(\pi)A_{1,\pi(1)}\cdots A_{n,\pi(n)}$$

Observe that this formula is precisely the same as the formula we are familiar with from calculus, except for the absolute value. To account for this, we choose local coordinates such that the determinant is always positive.

Definition. Same Orientation

The local coordinates $u^1,\ldots,u^n$ and $v^1,\ldots,v^n$ have the same orientation if $\det J>0$

Definition. Oriented Surface

A surface M is oriented if we choose one of the two classes for local coordinates with the same orientation.

Suppose we have two systems of coordinates. The change of coordinates matrix is nowhere degenerate because it is identity transformation. Thus, the determinant of the Jacobian matrix is nowhere zero as we move along points on the surface. So, by continuity, the determinant must be either positive at each point or negative at each point. Thus, all coordinates are divided into two classes, and coordinates belonging to the same class have the same orientation. By fixing one of these classes, we have an oriented surface.

Definition. Integrating over forms

Let M be oriented and $x^1, \ldots, x^n$ be positive oriented. Then for $\omega = f(x^1, \ldots, x^n)\, dx^1 \wedge \ldots \wedge dx^n$ we define

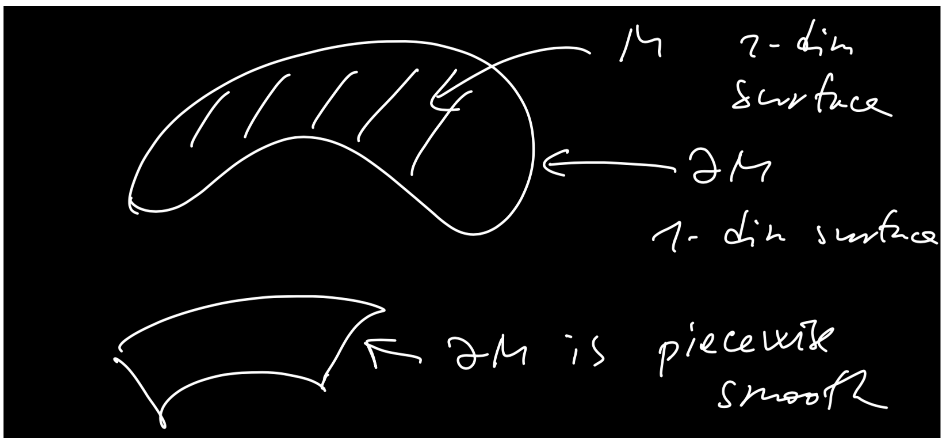
$$\int_M \omega = \int_M f(x^1, \ldots, x^n)\, dx^1 \wedge \ldots \wedge dx^n := \int f(x^1, \ldots, x^n)\, dx^1 \ldots dx^n$$

So we have defined it as the integral from calculus.

Why is the above definition reasonable? Because if we perform change of coordinates and our coordinates are positive definite then the form and the integral change in the same way.

4.5 Stokes' Formula

We consider smooth surfaces and boundaries



We also allow for the boundary ∂M to be piecewise smooth.

Theorem. Stokes' formula. Integrating over the differential of a form over a surface is the same as integrating the form over the boundary

$$\int_M d\omega = \int_{\partial M} \omega$$

We must discuss several concepts to help explain Stokes' formula. The first is the *restriction of a differential form.* Recall that the concept of a restriction of a function just involves restricting that function to some subset of its domain. Given $Y \subseteq X$ and $f : X \to \mathbb{R}$, the restriction is $f\big|_Y : Y \to \mathbb{R}$. Let us consider $\iota : Y \hookrightarrow X$, as the inclusion (or injection) map. Then, notice that

$$f\big|_Y = f \circ \iota = \iota^* f$$

Thus, the restriction is just the pullback of a function using the inclusion map. We can apply this concept of restriction to differential forms. Then, for $Y \overset{\iota}{\subset} X$ we have

$$\Omega^k(Y) \xleftarrow{\iota^*} \Omega^k(X)$$

where $\iota^*\omega := \omega\big|_Y \in \Omega^k(Y)$ and $\omega \in \Omega^k(X)$. This is important because we frequently use restrictions when we integrate functions.

Example. Consider an example from calculus. Suppose we are in $\mathbb{R}^2$ and we are integrating the circle, γ as below

$$\int_\gamma P\,dx + Q\,dy$$

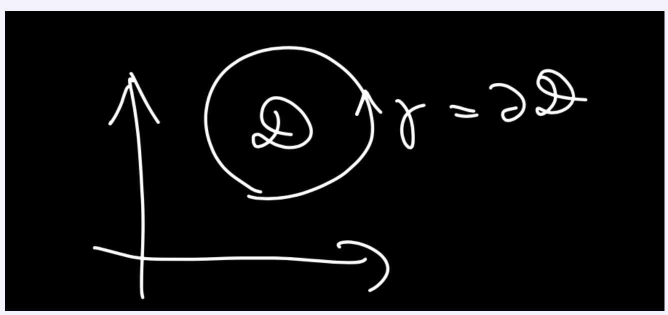
where $P\,dx + Q\,dy \in \Omega^1(\mathbb{R}^2)$. How is it possible that we are integrating a 1-form on $\mathbb{R}^2$? In fact, we are integrating the restriction. More generally, suppose that $Y \subseteq X$, $\omega \in \Omega^k(X)$ and $\dim Y = k$. Then we define

$$\int_Y \omega := \int_Y i^*\omega = \int_Y \omega\big|_Y$$

Thus, the expression involving the integral above means we are integrating the restriction of our form onto the circle. We need to consider the restriction because otherwise integrating over the curve does not have meaning: we would be integrating over a 1-form that is defined on $\mathbb{R}^2$ rather than on local coordinates of the curve.

1. We recall Green's formula.

Theorem. Green's formula. Suppose we have a region $\mathscr{D}$ in $\mathbb{R}^2$ and that its boundary is described by some closed curve $\gamma = \partial\mathscr{D}$ travelling counterclockwise.



The following equation holds

$$\oint_\gamma P\,dx + Q\,dy = \iint_{\mathscr{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$$

Let $\omega = P\,dx + Q\,dy$. We compute the differential

$$\begin{aligned} d\omega &= d(P\,dx + Q\,dy) \\ &= dP \wedge dx + P \wedge d^2x + dQ \wedge dy + Q \wedge d^2y \\ &= dP \wedge dx + dQ \wedge dy \\ &= \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy\right) \wedge dx + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy\right) \wedge dy \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy \end{aligned}$$

We obtain exactly the expression in the theorem. Remember that $\gamma = \partial\mathscr{D}$ as well. So, we see that Green's theorem is exactly a specific case of Stokes' formula in a planar domain.

2. We also know more versions of Stokes' formula in the form of **Gauss' divergence theorem**. Given some enclosed volume V and a boundary $\partial V = \Sigma$ which is some surface, we have

$$\iint_\Sigma P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy = \iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx \wedge dy \wedge dz$$
$$\iint_{\partial V} \omega = \iiint_V d\omega$$

3. Consider also the **Classical Stokes' formula** which is the case where we are in $\mathbb{R}^3$ and we have some surface Σ and some curve $\partial\Sigma = \gamma$.

4.6 More on Differential Forms

We may consider vector fields as vector-valued functions.

$$X \longleftrightarrow \begin{pmatrix} X^1 \\ \vdots \\ X^n \end{pmatrix}$$

in the standard basis for $\mathbb{R}^n$ and where $X^1, \dots, X^n$ are smooth functions.
We want to consider vector-valued forms. The idea is that we have a differential form $\omega(Y_1, \dots, Y_k)$ whose inputs are vector fields and whose output is a vector field. Recall that we usually obtain a function when plugging in vector fields into a differential form.

$$\omega \longleftrightarrow \begin{pmatrix} \omega^1 \\ \vdots \\ w^n \end{pmatrix}$$

where $\omega^i \in \Omega^k$.

Remark. Recall the formula

$$df = \frac{\partial f}{\partial x^i} dx^i$$

where we noted that du^i were the basis vectors of the dual space and $\frac{\partial f}{\partial x^i}$ the coefficients. Consider a basis $e_1, \dots, e_k$ in vector fields. Let $e^1 \dots, e^k$ be its dual, it is a basis in 1-forms. In the previous examples we discussed, we chose $e_1 = r_{u^1}, \dots, e_k = r_{u^k}$, which had the corresponding dual

$$e^1 = du^1, \dots, e^k = du^k$$

consisting of coordinate functionals. In certain situations, it may be more convenient to consider other bases. For instance, the basis above is not often orthonormal.

Proposition. We may write

$$df = e^i \cdot \partial_{e_i} f$$

Proof. Observe that

$$\begin{aligned}(e^i \cdot \partial_{e_i} f)(X) &= e^i(X) \cdot \partial_{e_i} f \\ &= X^i \cdot \partial_{e_i} f = \partial_X f = df(X)\end{aligned}$$

where X^i is the i^{th} coordinate of X in the basis $e_1, \dots, e_k$

□

We turn our attention back to the Gauss-Weingarten derivational formulae

$$\begin{aligned}\partial_{e_i} e_j &= \Gamma^p_{ij} e_p + b^\mu_{ij} \eta_\mu \\ \partial_{e_i} \eta_\nu &= -b^\mu_{ij} g_{\nu\mu} g^{jp} e_p + \kappa^\mu_{i,\nu} \eta_\mu\end{aligned}$$

Let us multiply both equations by e^i. Also, we denote

$$\boxed{\Gamma^p_j := e^i \Gamma^p_{ij} \text{ and } b^\mu_j := e^i b^\mu_{ij}}$$

Then, after multiplication and substitution, we have

$$de_j = \Gamma^p_j e_p + b^\mu_j \eta_\mu$$

This is an equality between vector-valued 1-forms. Notice that Γ^p_j and b^μ_j are 1-forms, and since we are multiplying them with vectors e_p and η_μ and summing, they are vector-valued 1-forms. The same holds for the second equation. We introduce the notation

$$\kappa^\mu_\nu := e^i \kappa^\mu_{i\nu}$$

and our equation is

$$d\eta_\nu = -b^\mu_j g_{\nu\mu} g^{jp} e_p + \kappa^\mu_\nu \eta_\mu$$

Recall the property that $d^2 = 0$. Let us apply d to the equation involving de_j. We have

$$\begin{aligned} 0 = d^2 e_j &= d(\Gamma^p_j e_p + b^\mu_j \eta_\mu) \\ &= (d\Gamma^p_j e_p - \Gamma^p_j \wedge de_p) + (db^\mu_j \eta_\mu - b^\mu_j \wedge d\eta_\mu) \end{aligned}$$

Observe that we can find de_p and $d\eta_\mu$ using the equations we just derived above. However, we would like to look only at the tangent part. We can conclude that both the tangent and normal components are 0 because they are orthogonal and their sum is 0.

$$\begin{aligned} 0 &= d\Gamma^p_j e_p - \Gamma^p_j \wedge de_p - b^\mu_j \wedge d\eta_\mu \\ &= d\Gamma^p_j e_p - \Gamma^p_j \wedge \Gamma^q_p e_q + b^\mu_j \wedge b^\alpha_q g_{\mu\alpha} g^{qp} e_p \\ &= d\Gamma^r_j e_r - \Gamma^p_j \wedge \Gamma^r_p e_r + b^\mu_j \wedge b^\alpha_q g_{\mu\alpha} g^{qr} e_r \\ &= (d\Gamma^r_j - \Gamma^p_j \wedge \Gamma^r_p + b^\mu_j \wedge b^\alpha_q g_{\mu\alpha} g^{qr}) e_r \end{aligned}$$

Since this vector is equal to 0 and e_r forms a basis, we know that each coordinate is 0

$$0 = d\Gamma^r_j - \Gamma^p_j \wedge \Gamma^r_p + b^\mu_j \wedge b^\alpha_q g_{\mu\alpha} g^{qr}$$

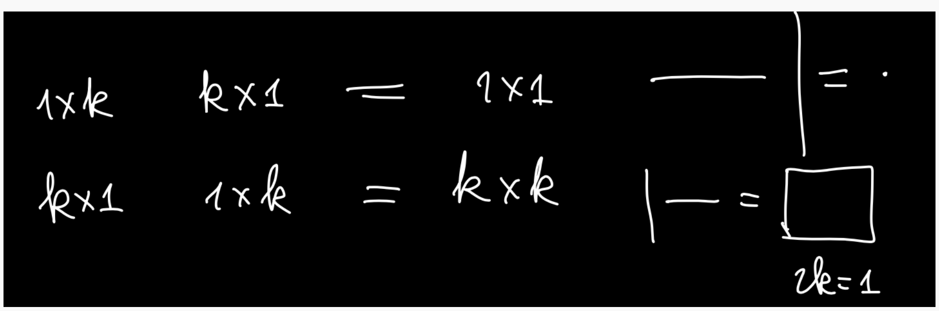
This is an identity for 2-forms. By rearrangement, we have

$$d\Gamma^r_j + \Gamma^r_p \wedge \Gamma^p_j = b^\alpha_q g_{\mu\alpha} g^{qr} \wedge b^\mu_j$$

Aside. Consider the formula for matrix multiplication

$$(AB)^i_j = A^i_k B^k_j$$

These operations are possible whenever we have multiplication and addition, as in the structure of a ring. So, we may think of matrices with entries belonging to some ring, i.e. $\Omega(\Sigma)$. Further, note that



Let us consider a matrix Γ of 1-forms with elements Γ^i_j. Given this consideration, we may rewrite

$$(d\Gamma + \Gamma \wedge \Gamma)^r_j$$

We note that we use $\wedge$ instead of multiplication because the elements of the matrix are differential forms. On the left, we have a $k \times k$ matrix consisting of 2-forms, where k is the dimension of the surface Σ.

Considering the picture from the aside, we want to produce a $k \times k$ matrix on the right-hand side using $b^\alpha_q g_{\mu\alpha} g^{qr} \wedge b^\mu_j$. Let us consider b^μ_j as the elements of a $1 \times k$ row

$$b^\mu = (b^\mu_1, \ldots, b^\mu_k)$$

Let us consider $b^\alpha_q g_{\mu\alpha} g^{qr}$ as the elements of a $k \times 1$ column

$$b_\mu = \begin{pmatrix} b^\alpha_q g_{\mu\alpha} g^{q1} \\ \vdots \\ b^\alpha_q g_{\mu\alpha} g^{qk} \end{pmatrix}$$

Based on this interpretation, we may write

$$\boxed{(d\Gamma + \Gamma \wedge \Gamma)^r_j = (b_\mu \wedge b^\mu)^r_j}$$

This is called the **Gauss equation**. In fact, this is called the Riemann curvature tensor, but we are not able to discuss this in detail. So, we have recovered the matrices based on previous equation, which we may understand as describing the elements of the matrix.

Let us consider a special case of a 2-dimensional surface in Euclidean space, $\mathbb{R}^3$, ($k = 2$, $n = 3$). Consider an orthonormal basis e_1, e_2 in the tangent vector fields and the basis $\eta_1 = m$ the unit normal vector. Then, we have

$$g_{ij} = \langle e_i, e_j \rangle = \delta_{ij}$$
$$g_{\nu\nu} = \langle \eta_\nu, \eta_\nu \rangle = 1$$

The equations above follow from orthonormality of the vectors.

Let us consider the e_{12} element of the Gauss equation

$$\begin{aligned} \left(\sum_\mu b_\mu \wedge b^\mu \right)^1_2 &= (b_{\bar{1}} \wedge b^{\bar{1}})^1_2 \\ &= b^{\bar{1}}_q g_{\bar{1}\bar{1}} g^{q1} \wedge b^{\bar{1}}_2 \\ &= b^{\bar{1}}_q \cdot 1 \cdot \delta^{q1} \wedge b^{\bar{1}}_2 \\ &= b^{\bar{1}}_1 \wedge b^{\bar{1}}_2 \end{aligned}$$

Recall that $b^\mu_i = e^j b^\mu_{ji}$. By substitution

$$b^{\bar{1}}_1 \wedge b^{\bar{1}}_2 = (e^1 b^{\bar{1}}_{11} + e^2 b^{\bar{1}}_{21}) \wedge (e^1 b^{\bar{1}}_{12} + e^2 b^{\bar{1}}_{22})$$

Recall also that we proved

$$B(X, Y) = \mathrm{II}(X, Y) \cdot \vec{m}$$

From this equality, we see that $b^{\bar{1}}_{11} = L$, $b^{\bar{1}}_{12} = b^{\bar{1}}_{21} = M$, and $b^{\bar{1}}_{22} = N$. Substituting again, we see

$$\begin{aligned}(e^1 b^{\bar{1}}_{11} + e^2 b^{\bar{1}}_{21}) \wedge (e^1 b^{\bar{1}}_{12} + e^2 b^{\bar{1}}_{22}) &= (e^1 L + e^2 M) \wedge (e^1 M + e^2 N) \\ &= LN \cdot e^1 \wedge e^2 + e^2 \wedge e^1 \cdot M^2 \\ &= (LN - M^2) \cdot e^1 \wedge e^2 \\ &= \frac{\det \mathrm{II}}{\det \mathrm{I}} \cdot e^1 \wedge e^2 = K e^1 \wedge e^2\end{aligned}$$

Recall that we derived a formula which describes Christoffel symbols in terms of the metric.

$$\Gamma^p_{ij} = \frac{1}{2} g^{pq} \left(\frac{\partial g_{qj}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^q} + \frac{\partial g_{iq}}{\partial u^j} \right)$$

Notice that Γ^p_{ij} depends on metric that only involves the first fundamental form. So this means that in the Gauss equation, the left-hand side $d\Gamma + \Gamma \wedge \Gamma$ depends only on the first fundamental form. Thus, the right-hand side, which we have derived as the Gaussian curvature K, also depends only on the first fundamental form.

We arrive now to the Gauss' Theorema egregium.

Theorem. The Gaussian curvature K is completely defined by the first fundamental form.

Previously, we defined Gaussian curvature as the product of principal curvatures, which themselves depend on the first and second fundamental form. From our result above, we see that it only depends on the first fundamental form.

To determine the explicit formula, we consider

$$\begin{aligned}K e^1 \wedge e^2 &= (d\Gamma + \Gamma \wedge \Gamma)^1_2 \\ &= d\Gamma^1_2 + \Gamma^1_i \wedge \Gamma^i_2\end{aligned}$$

However, this holds only in an orthonormal basis.

4.7 Simple Case of Gauss-Bonnet

Consider a surface with a closed loop γ that encloses a region $\mathcal{D}$. We want to determine the angle of rotation α resulting from the parallel transport of a vector along this closed loop

Using Gauss' theorem egregium and the relationship we have discovered between the Gaussian curvature and the first fundamental form, we want to show the equation

$$\alpha = \iint_{\mathscr{D}} K\, dA$$

We know that

$$dA = \sqrt{EF - F^2}\, du \wedge dv$$

If we have e_1, e_2 as an orthonormal basis, then we have

$$dA = \sqrt{\det \mathrm{I}} \cdot e^1 \wedge e^2 = e^1 \wedge e^2$$

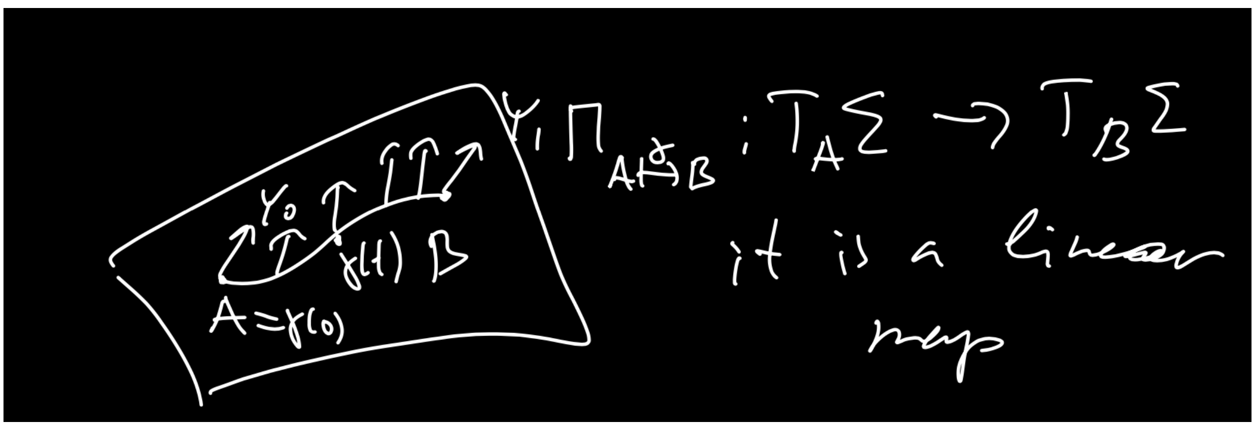
Thus, for an orthonormal basis, we have on the right-hand side

$$\iint_{\mathscr{D}} K\, dA = \iint_{\mathscr{D}} K\, e^1 \wedge e^2$$

To interpret the angle α, we remark that it is in fact the integral of the expression we obtained in the preceding section. By substitution for $K\, e^1 \wedge e^2$, we have:

$$\alpha = \iint_{\mathscr{D}} d\Gamma_2^1 + \Gamma_i^1 \wedge \Gamma_2^i$$

We return to parallel transport.



We have a points A, B on some curve γ, which lies on a surface Σ. We have an initial tangent vector Y_0 at $A = \gamma(0)$ and a parallel vector field with these initial conditions. We have some final vector Y_1, the result of the parallel transport. Given this setup, we can consider a map

$$\Pi_{A \overset{\gamma}{\mapsto} B} : T_A\Sigma \to T_B\Sigma$$

This is a linear map because the equation of parallel transport is a linear equation. We can also think of the preservation of lengths and angles (i.e. if we multiply the initial vector by some scalar, the end result is also scaled by the same amount). Now denote

$$\Pi(t) := \Pi_{A \overset{\gamma}{\mapsto} \gamma(t)} : T_A\Sigma \to T_{\gamma(t)}\Sigma$$

Consider a basis $e_1, \ldots, e_k$ in the tangent vector fields. We have

$$\begin{aligned} Y(t) &= \Pi(t)Y_0 \\ Y^i(t) &= \Pi^i_j(t)Y^j_0 \end{aligned} \tag{1}$$

where Π^i_j is a matrix. Above, Y_0 acts as an input to the map specified by $\Pi(t)$.
For the above equations, we consider the Cauchy problem for parallel transport. Recall that the Cauchy problem for parallel transport has the following form:

$$\begin{cases} \nabla_{\dot\gamma} Y(t) = 0 \\ \quad Y(0) = Y_0 \end{cases}$$

and in coordinates it is

$$\begin{cases} \dot Y^i(t) + \Gamma^i_{pq}\dot u^p Y^q(t) = 0 \\ \qquad\qquad\qquad Y^i(0) = Y^i_0 \end{cases}$$

Let us plug Equation (1) into the above system, we obtain

$$\begin{cases} \dot\Pi^i_j(t)Y^j_0 + \Gamma^j_{pq}\dot u^p \Gamma^q_j(t)Y^j_0 = 0 \\ \qquad\qquad\qquad \Pi^i_j(0)Y^j_0 = Y^i_0 \end{cases}$$

For the first equation, we can divide through by the constant Y^j_0 and for the second equation, we can deduce that it is the Kronecker delta

$$\begin{cases} \dot\Pi^i_j(t) + \underbrace{\Gamma^j_{pq}\dot u^p}_{=\Gamma^i_q(\dot u)} \Gamma^q_j(t) = 0 \\ \qquad\qquad \Pi^i_j(0) = \delta^i_j \end{cases}$$

Note that

$$\Gamma^i_q(\dot u) = (e^p\Gamma^j_{pq})(\dot u) = \Gamma^j_{pq} \cdot e^p(\dot u^1 e_1 + \ldots \dot u^k e_k) = \Gamma^j_{pq}\dot u^p$$

The above is the equation for the operator of parallel transport. Finally, we may write

$$\begin{cases} \quad \dot\Pi = -\Gamma(\dot u)\Pi \\ \Pi(0) = \mathrm{Id} \end{cases}$$

Aside. We consider a rudimentary ODE

$$\begin{cases} \quad \dot y = -a(t)y \\ y(0) = y_0 \end{cases}$$

Using separation of variables, we see that

$$\frac{dy}{dt} = -ay \implies \int \frac{dy}{y} = -\int a\,dt \implies \ln(y) = -\int a\,dt$$

So the final solution is

$$y(t) = y_0 e^{-\int_0^t a(t)\,dt} \tag{2}$$

The solution in the aside holds because the following equation holds

$$\frac{d}{dt}(e^{A(t)}) = \dot{A}e^{A(t)}$$

Notice that this works for scalars, but not in general for matrices. Recall that

$$\begin{aligned} e^A &= I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \dots \\ \frac{d}{dt}(e^A) &= 0 + \dot{A} + \frac{1}{2!}(\dot{A}A + A\dot{A}) + \dots \end{aligned}$$

Notice that if we have numbers, then the expression commutes and $\dot{A}A + A\dot{A} = 2\dot{A}A$, but this equality does not hold generally for matrices.

Observation. Orthogonal and skew-symmetric matrices of dimension 2×2 commute.

Example. We compute

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} = \begin{pmatrix} t\sin\alpha & t\cos\alpha \\ -t\cos\alpha & t\sin\alpha \end{pmatrix} = \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

If e_1, e_2 is an orthonormal basis, then

$$\Gamma_{ij}^p = -\Gamma_{ip}^j$$

Recall the following justification, since the basis is orthonormal, we have $\langle e_j, e_p \rangle = \delta_{jp}$ and

$$\begin{aligned} \partial_{e_i} \langle e_j, e_p \rangle &= 0 \\ \langle \nabla_{e_i} e_j, e_p \rangle + \langle e_j, \nabla_{e_i} e_p \rangle &= 0 \\ \left\langle \Gamma_{ij}^q e_q, e_p \right\rangle + \left\langle e_j, \Gamma_{ip}^q e_q \right\rangle &= 0 \\ \Gamma_{ij}^p + \Gamma_{ip}^q &= 0 \end{aligned}$$

Based on this, we have

$$\Gamma_{ij}^p = -\Gamma_{ip}^j \iff \Gamma_j^p = -\Gamma_p^j \iff \Gamma^T = -\Gamma$$

This shows that the matrix Γ inside the equation $\dot{\Pi} = -\Gamma(\dot{u})\Pi$ is skew-symmetric. As a result, we may apply formula (2) to obtain the matrix exponential

$$\Pi(t) = \exp\left(-\int_0^t \Gamma(\dot{u})\,dt\right) = e^{-\int_0^t \Gamma}$$

Since Γ is skew-symmetric, it may be represented as

$$\Gamma = \begin{pmatrix} 0 & \Gamma_2^1 \\ -\Gamma_2^1 & 0 \end{pmatrix}$$

Now, we continue our computation to see

$$\Pi(t) = \exp \int_0^t \begin{pmatrix} 0 & -\Gamma_2^1 \\ \Gamma_2^1 & 0 \end{pmatrix} = \exp \begin{pmatrix} 0 & -\int_0^t \Gamma_2^1 \\ \int_0^t \Gamma_2^1 & 0 \end{pmatrix}$$

Aside. From linear algebra, we know that

$$\exp \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix} = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}$$

To see this, consider that

$$\begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix}\begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix} = \begin{pmatrix} -x^2 & 0 \\ 0 & -x^2 \end{pmatrix}$$
$$\begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix}\begin{pmatrix} -x^2 & 0 \\ 0 & -x^2 \end{pmatrix} = \begin{pmatrix} 0 & x^3 \\ -x^3 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix}^4 = \begin{pmatrix} -x^2 & 0 \\ 0 & -x^2 \end{pmatrix} = \begin{pmatrix} x^4 & 0 \\ 0 & x^4 \end{pmatrix}$$

Thus, it follows that

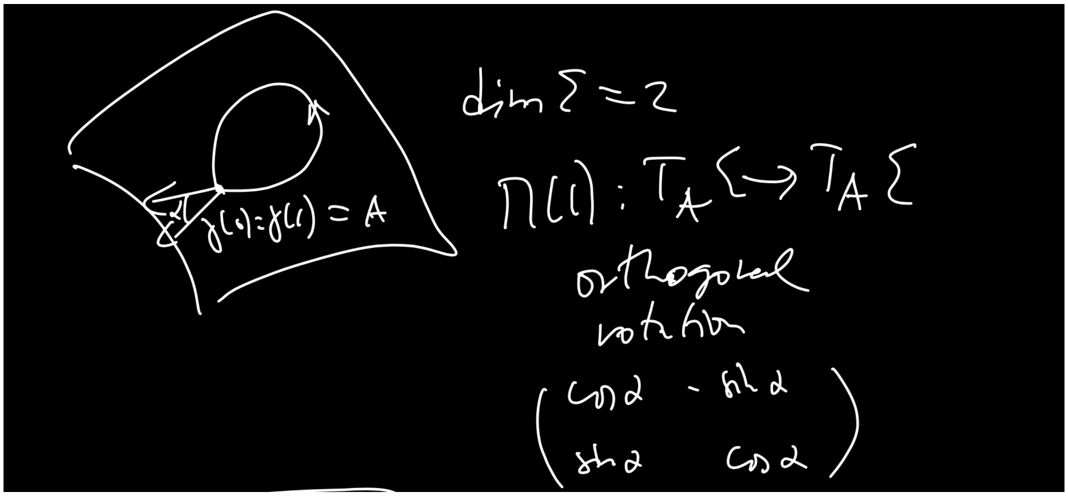
$$\exp \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix} = \begin{pmatrix} 1 - \frac{x^2}{2!} \pm \dots & -x + \frac{x^3}{3!} \pm \dots \\ x - \frac{x^3}{3!} \pm \dots & 1 - \frac{x^2}{2!} \pm \dots \end{pmatrix}$$

and our asserted equality follows.

Using the aside, we conclude that

$$\exp \begin{pmatrix} 0 & -\int_0^t \Gamma_2^1 \\ \int_0^t \Gamma_2^1 & 0 \end{pmatrix} = \begin{pmatrix} \cos \int_0^t \Gamma_2^1 & -\sin \int_0^t \Gamma_2^1 \\ \sin \int_0^t \Gamma_2^1 & \cos \int_0^t \Gamma_2^1 \end{pmatrix}$$

Now, consider the setup below. If we have $\Pi(1)$, then the result must be a rotation because of parallel transport preserving lengths and angles, and since $\Pi(1)$ maps the original vector into the same tangent plane. Since $\Pi(0)$ has determinant 1, as the identity matrix and $\Pi(t)$ is continuous, the determinant remains 1. So, the transformation is an orthogonal rotation.



Finally, comparing the operator we have derived with the fact that parallel transport along a loop results in an orthogonal rotation, we see that

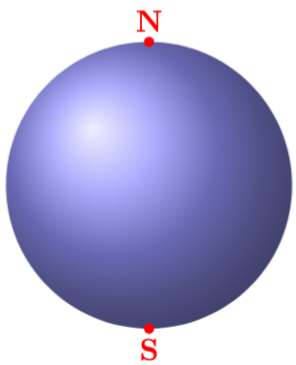
$$\alpha = \oint_\gamma \Gamma^1_2$$

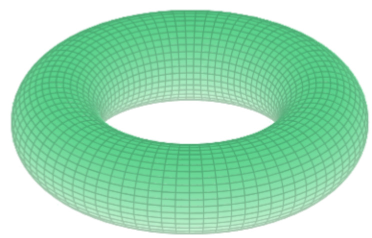
5 Special Topics

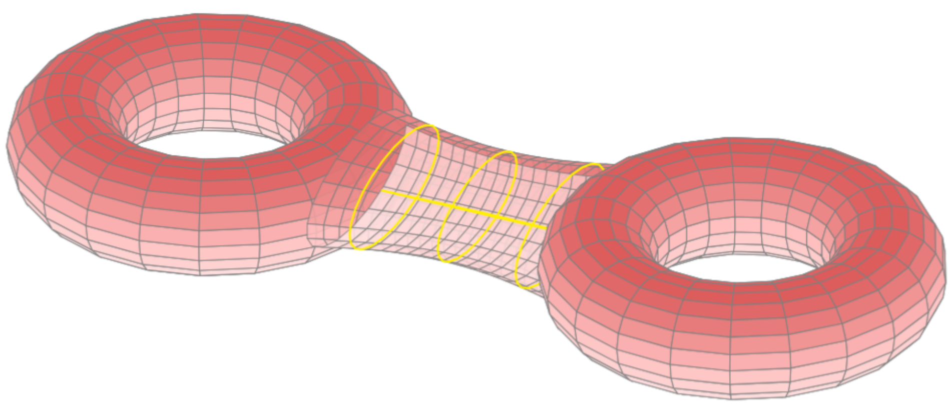
Placeholder: Lecture 12 covers Triangulation of Surfaces and Euler characteristic. Lecture 13 and the end of Lecture 12 cover the idea of Variation. We will fill in these notes later.

5.1 Smooth Manifolds

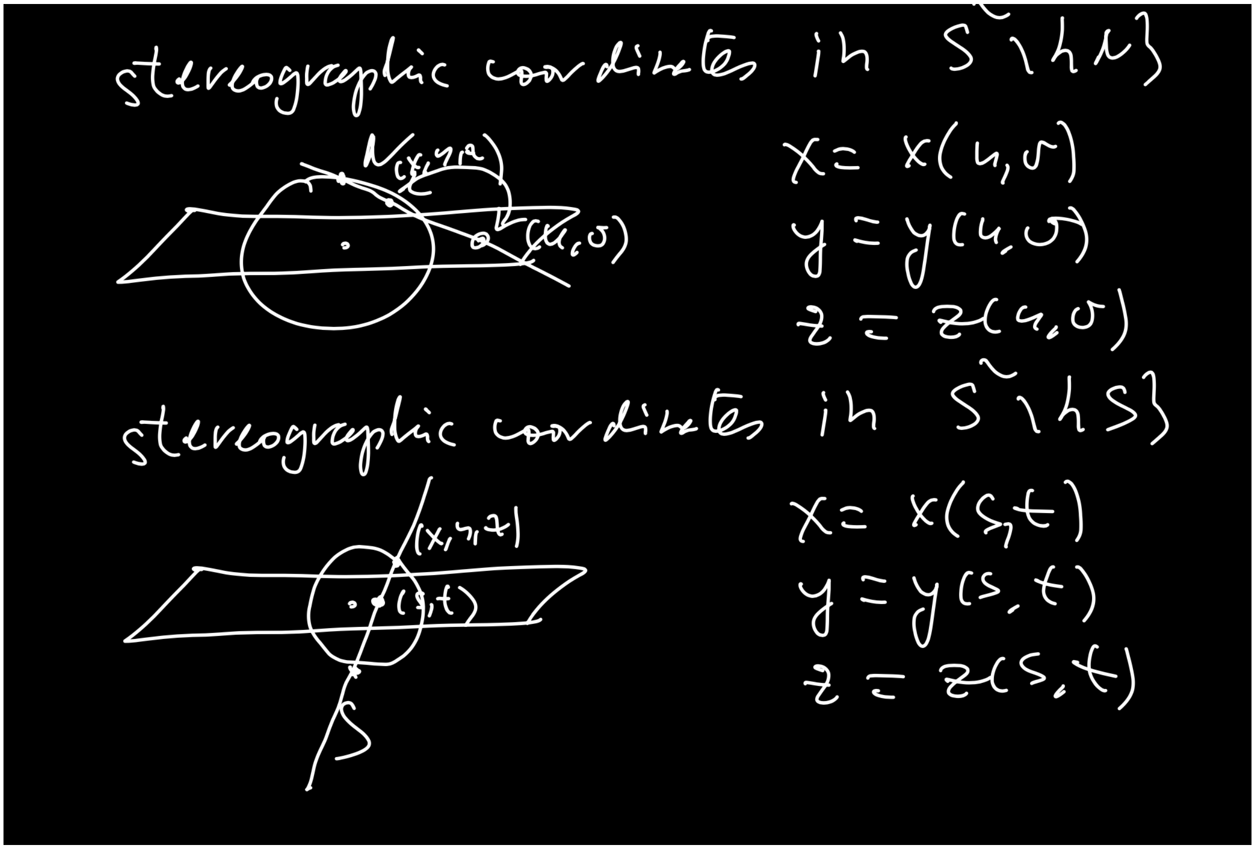
We have some examples of smooth manifolds below. The sphere, the torus, and the double torus (informally, the pretzel).





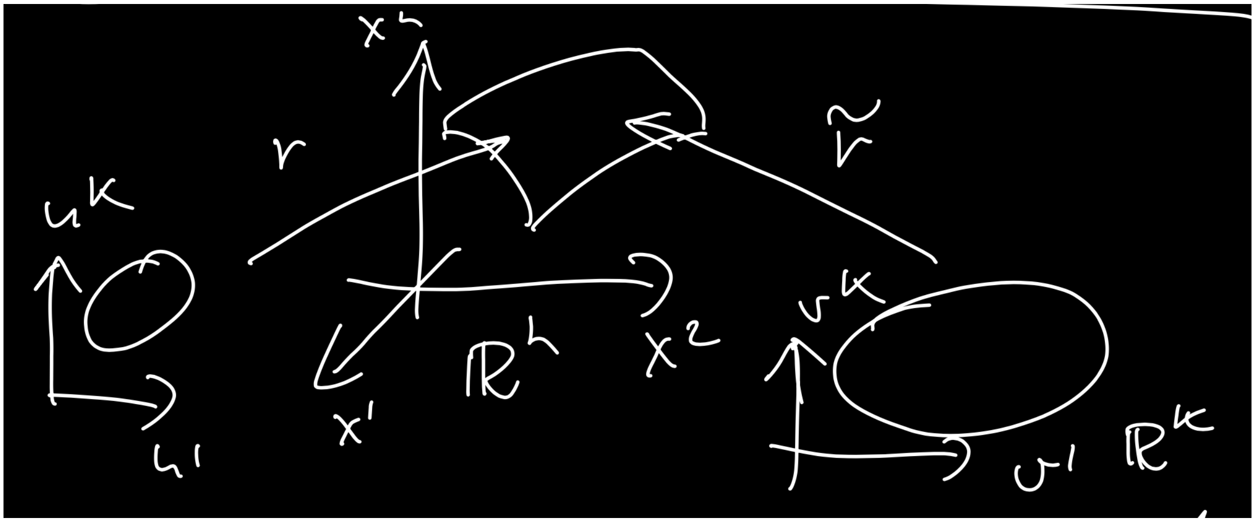


These are examples relevant to our prior discussion of the Gauss-Bonnet formula. However, it turns out that the setting of smooth, regular, parameterized surfaces is not very convenient. This inconvenience results from a lack of a global parameterized description. For instance, when we compute the integral, we have to cut these shapes into pieces. The sphere is cut into the Northern and Southern hemisphere during such a computation. This motivates the definition of a manifold, which we will proceed to in parts. Recall the idea of stereographic projection and stereographic coordinates, shown in the diagrams on the next page.



The idea here is we associate every point on the sphere with some point on the equatorial plane, based on either a line which passes through the North pole, or the South pole. If we consider this process of association, we note that we get a 1-1 correspondence for all points on the sphere, except the Northern pole. In the above description, we see that our sphere is covered by two maps, what will be called *charts*. In both of these charts we have local coordinates, (u, v) for one and (s, t) for the other. A change of (local) coordinates exists, for instance $u = u(s, t)$, $v = v(s, t)$. So, in this example, we see that when we have a surface, we may consider covering the surface with parametrized surfaces. To be explicit, S^2 is covered by the parametrized surfaces $S^2 \setminus \{N\}$ and $S^2 \setminus \{S\}$. At a more basic level, the reparameterization of a curve is also a change of local coordinates.

With the concrete example of stereographic coordinates which describe parameterized surfaces covering the sphere in mind, we now consider a more general situation. In this setup, we begin with a k-dimensional surface in $\mathbb{R}^n$ (in the sense of the shape, not the map). Further, we have two systems of local coordinates which parameterize the surface, $(u^1, \dots, u^k)$ and $(v^1, \dots, v^k)$. These coordinates are mapped by $r, \tilde{r}$, which are smooth, regular, k-dimensional surfaces in $\mathbb{R}^n$.



Suppose that the images of r and $\tilde{r}$ coincide. Then, it is possible to express one coordinate in terms of the other. Since r is regular, the rank of the Jacobian matrix is k. We may write

$$\begin{aligned} u^1 &= u^1(x^1, \ldots, x^k) \\ &\vdots \\ u^k &= u^k(x^1, \ldots, x^k) \end{aligned}$$

by the inverse function theorem. Since the surface is expressible in terms of local coordinates in v, we see that

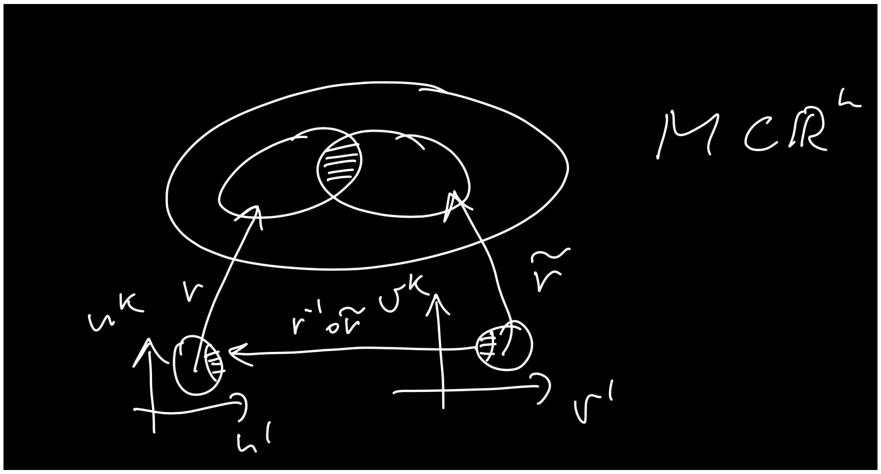
$$\begin{aligned} u^1 &= u^1(x^1(v^1, \ldots, v^k), \ldots, x^k(v^1, \ldots, v^k)) \\ &\vdots \end{aligned}$$

Hence, there always exists a *smooth* change of coordinates (the above is a composition of smooth functions).

Definition. Smooth Manifold in $\mathbb{R}^n$

A smooth manifold of dimension k embedded in $\mathbb{R}^n$ is a subset $M \subseteq \mathbb{R}^n$ such that for any point $p \in M$, there exists a neighborhood U such that $p \in U \subseteq M$ and a a regular, smooth parametrized surface $r : V \to U$ (where $V \subseteq \mathbb{R}^k$) which is a one-to-one correspondence.

Using our example from before, if we pick a point on the sphere that is not one of the poles, we can just consider the stereographic coordinate to get the surface from $\mathbb{R}^n$ to some neighborhood of that point on the sphere. If the point is one of the poles, say the North pole, we should consider the map from $\mathbb{R}^2$ to $S^2 \setminus \{S\}$ instead.



Above, we see the abstract idea. Our manifold M is covered by some parameterized surfaces, and on the sets where the surfaces intersect, regularity guarantees that a change of coordinates exists. This change of coordinates, from v to u is given by $r^{-1} \circ \tilde{r}$. Further, the parameterized surfaces indicated by U in the definition are often called *coordinate patches*.

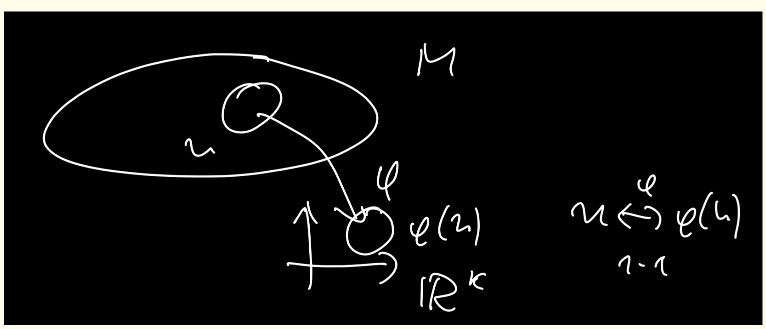
Now, we move to an even more general topological setting.

Definition. Manifold

An (abstract) manifold M is a Hausdorff, second countable topological space.

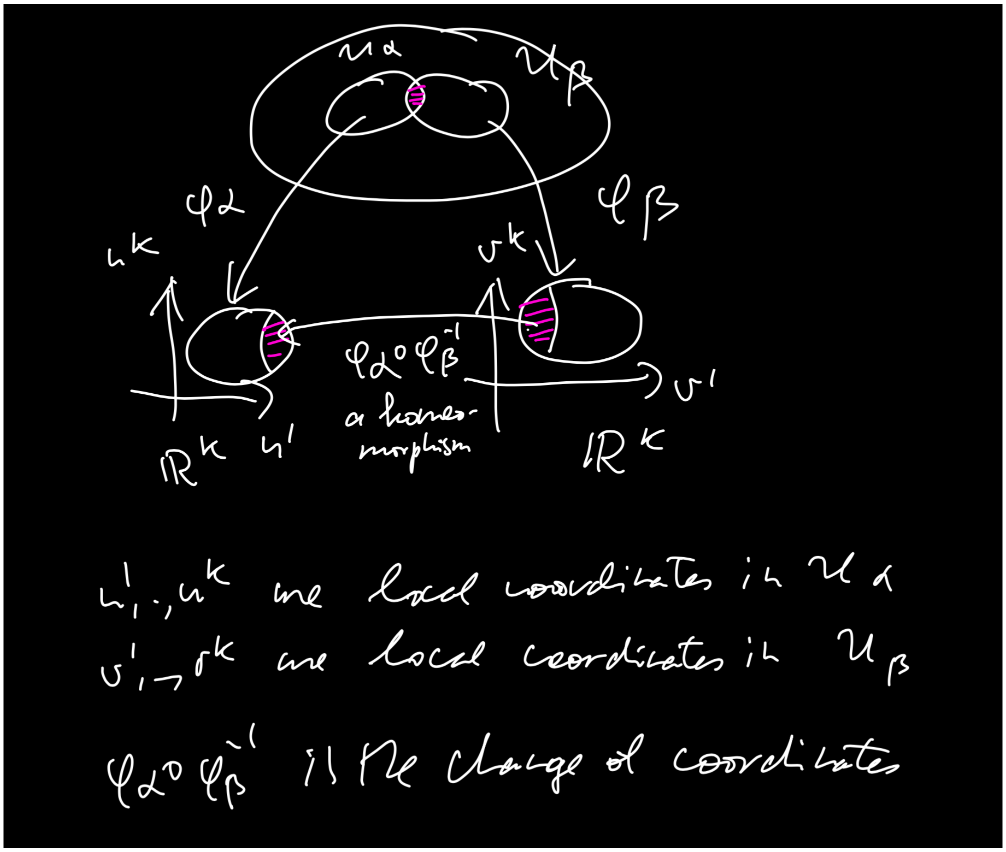
Definition. Coordinate Chart

A coordinate chart is (U, φ), where U is an open subset of M and $\varphi : U \to \mathbb{R}^k$ is a homeomorphism.



The difference between this definition and the example given for surfaces is that the map is in the other direction. However, this is not an issue because it is a homeomorphism, so the continuous inverse exists.

When we have multiple coordinate charts, there exists a change of coordinates between sections on the manifold where these coordinate charts overlap. The situation is similar to that of the smooth manifold in $\mathbb{R}^n$. We show a diagram of this general setting.



One difference between our situation in $\mathbb{R}^n$ and this abstract situation is that we cannot say anything about smoothness. We have only continuity since the functions are homeomorphisms.

Definition. Compatible

We say that charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are $\mathscr{C}^k$-compatible if $\varphi_\alpha \circ \varphi_\beta^{-1} : \mathbb{R}^k \to \mathbb{R}^k$ is $\mathscr{C}^k$-smooth.

Definition. Atlas

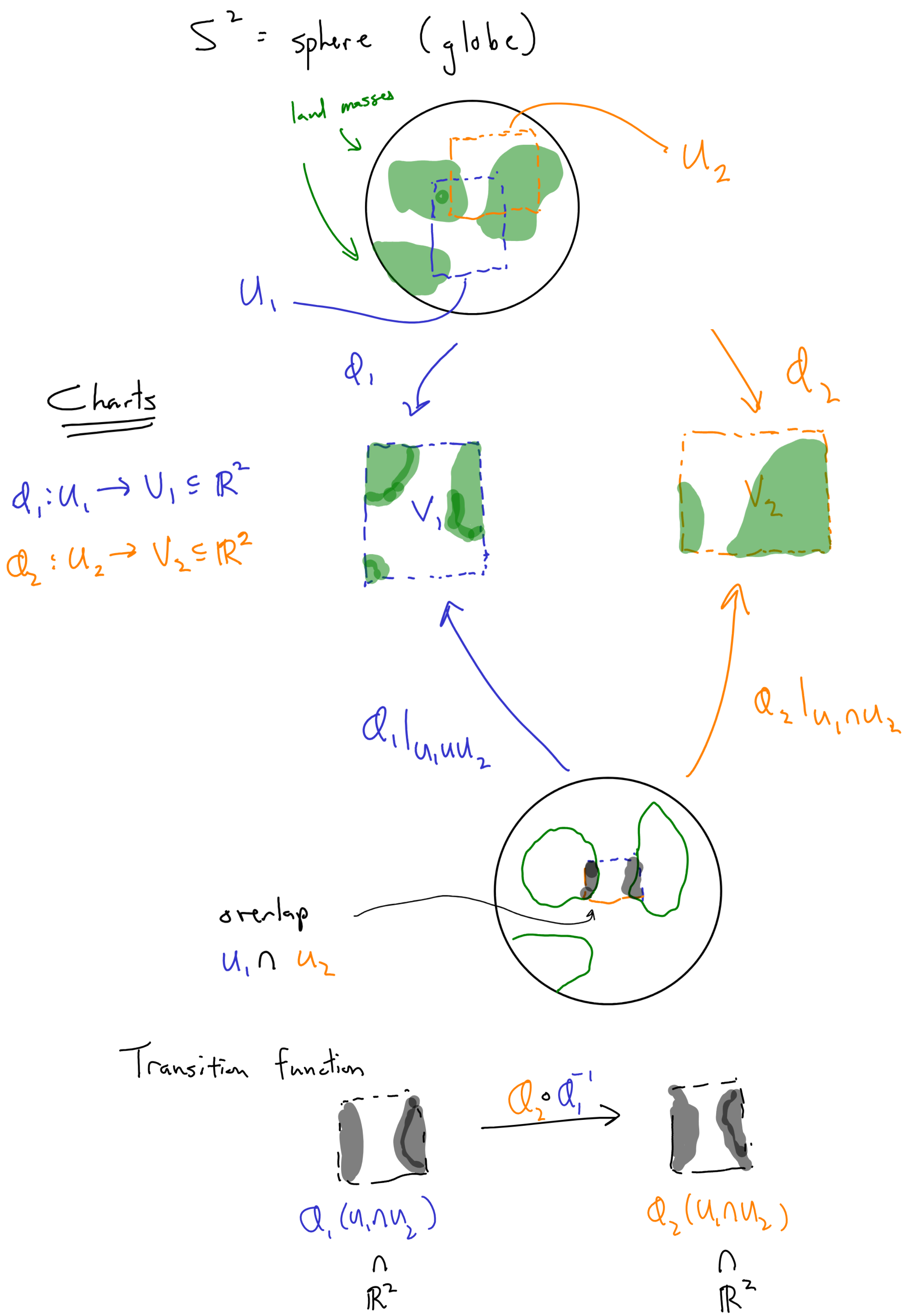
The collection of charts $A = \{(U_\alpha, \varphi_\alpha)\}$ is called an atlas of class $\mathscr{C}^k$ if

1. The charts cover the manifold M, that is

$$\bigcup_\alpha U_\alpha = M$$

2. For all α, β, the charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are $\mathscr{C}^k$-compatible.

One example of an atlas is its namesake, the globe.


S^2 = sphere (globe)
land masses
U_2
U_1
φ_1
φ_2
Charts
φ_1 : U_1 → V_1 ⊆ ℝ^2
φ_2 : U_2 → V_2 ⊆ ℝ^2
V_1
V_2
φ_1|_{U_1∩U_2}
φ_2|_{U_1∩U_2}
overlap
U_1 ∩ U_2
Transition function
φ_2 ∘ φ_1^{-1}
φ_1(U_1∩U_2)
∩
ℝ^2
φ_2(U_1∩U_2)
∩
ℝ^2

Definition. Smooth Manifold

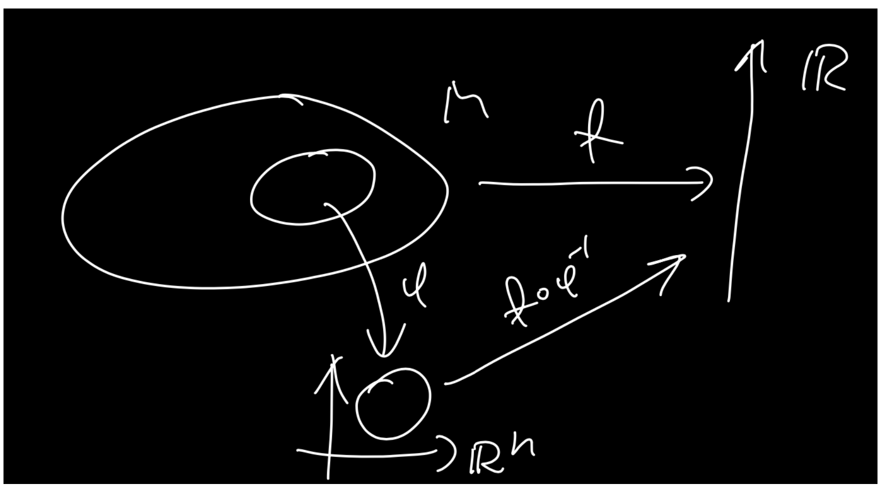
A smooth manifold is a Hausdorff second-countable topological space M with an atlas.

Definition. Equivalence of Atlases

We say that atlases A_1 and A_2 are equivalent if $A_1 \cup A_2$ is also an atlas. This condition is really saying that the charts from both atlases are all compatible.

The above definition provides us with equivalence classes of atlases. It turns out that on the same M, it is possible to have atlases that do not belong to the same equivalence classes. The same set can have different smooth structures.

We now consider smooth functions on a manifold M of class $\mathscr{C}^k$, where $\dim M = n$. A problem that we face is that M is some arbitrary topological. While we are able to say that f is continuous since we have the structure of the topology, we cannot say more about smoothness (i.e. differentiability).



Referring to the picture above, we see that we can use the coordinate chart to get a composition $f \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}$, which allows us to work with smoothness in the context of real spaces.

Definition. Smooth function

A function f is smooth of class $\mathscr{C}^l$, $l \leq k$ if for all charts (U, φ) we have

$$f \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R} \quad \text{is } \mathscr{C}^l$$

Now, there is a question of whether this definition makes sense (i.e. does not lead to a contradiction). Resume at Lecture 14, 39:00