Complex Analysis Notes

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Definition. *Complex Numbers*
Complex numbers are given by the set

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

Often, we will denote $z = a + bi$, with $a = \text{Re}(z)$ and $b = \text{Im}(z)$. Also, $i^2 = -1$ and $i = \sqrt{-1}$. Note that $(\mathbb{C}, +, \cdot)$ is a field which contains the real field.

Operations on Complex Numbers

1. *Addition of complex numbers.*

$$z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$$

2. *Multiplication of complex numbers.*

$$\begin{aligned} z_1 \cdot z_2 &= (a_1 + ib_1)(a_2 + ib_2) = a_1a_2 + ia_1b_2 + ib_1a_2 + i^2b_1b_2 \\ &= (a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2) = z_2 \cdot z_1 \end{aligned}$$

3. For all $z \in \mathbb{C}$, we have

$$\begin{aligned} 0 &= 0 + 0 \cdot i, \quad 0 + z = z \\ 1 &= 1 + 0 \cdot i, \quad 1 \cdot z = z \\ -z &= -(a + bi) = (-a) + (-b)i, \quad z + (-z) = 0 \end{aligned}$$

4. *Division of complex numbers.* Suppose that $z = a + bi \neq 0$. Note that this is equivalent to assuming $a \neq 0 \vee b \neq 0$.

$$z^{-1} = \frac{1}{z} = \frac{1}{a + bi} = \frac{a - bi}{(a + bi)(a - bi)} = \frac{a - bi}{a^2 + b^2}$$

5. The conjugate. Conjugates are symmetric with respect to the real axis.

$$\overline{z} = \overline{a + bi} = a - bi, \quad z \cdot \overline{z} = \overline{z} \cdot z = a^2 + b^2 = |z|^2$$

Fact. $\mathbb{C}$ is an *algebraically closed* field. Every polynomial with $\mathbb{C}$-coefficients always has a root in $\mathbb{C}$. This is equivalent to saying that polynomials with complex coefficients always split. The number of roots, accounting for multiplicity, is equal to the degree of the polynomial. Given a polynomial

$$P(z) = a_0z^n + a_1z^{n-1} + \ldots + a_{n-1}z + a_n$$

there exists $w \in \mathbb{C}$ such that $P(w) = 0$. Notice how this is not true for $\mathbb{R}$, if we consider $x^2 + 1 = 0$.

1.1 Discussion on the Polar Form

We can also give complex numbers in the so-called *polar form.*

$$z = a + bi = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = r \cdot e^{i\theta}$$

This gives a geometric interpretation of the multiplication complex numbers as involving the addition of the *argument* θ and the multiplication of the *modulus* r. Observe that $i^2 = -1,\ \ i^3 = -i,\ \ i^4 = (-1)^2 = 1$ may be thought of as successive rotations.

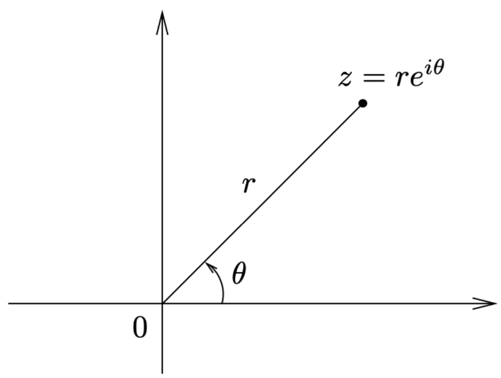


Figure 2. The polar form of a complex number

Above, the point would be located at (a, b). This form has a connection to **Euler's formula**.

Aside. Recall that Euler's formula states

$$e^{i\theta} = \cos\theta + i\sin\theta$$

A special case of this formula is $e^{i\pi} = -1$. Considering the Taylor expansion leads to this formula. Recalling that $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$, we have

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \dots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right) \\ &= \cos\theta + i\sin\theta \end{aligned}$$

Returning to our geometric interpretation, we have

$$z_1 z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = (r_1 r_2)e^{i(\theta_1+\theta_2)}$$

which shows why the lengths are multiplied and the angles are added. Thinking of z_1 and z_2 as vectors, we can also see that $z_1 + z_2$ and $z_1 - z_2$ are the diagonals of the parallelogram formed by z_1 and z_2. We have

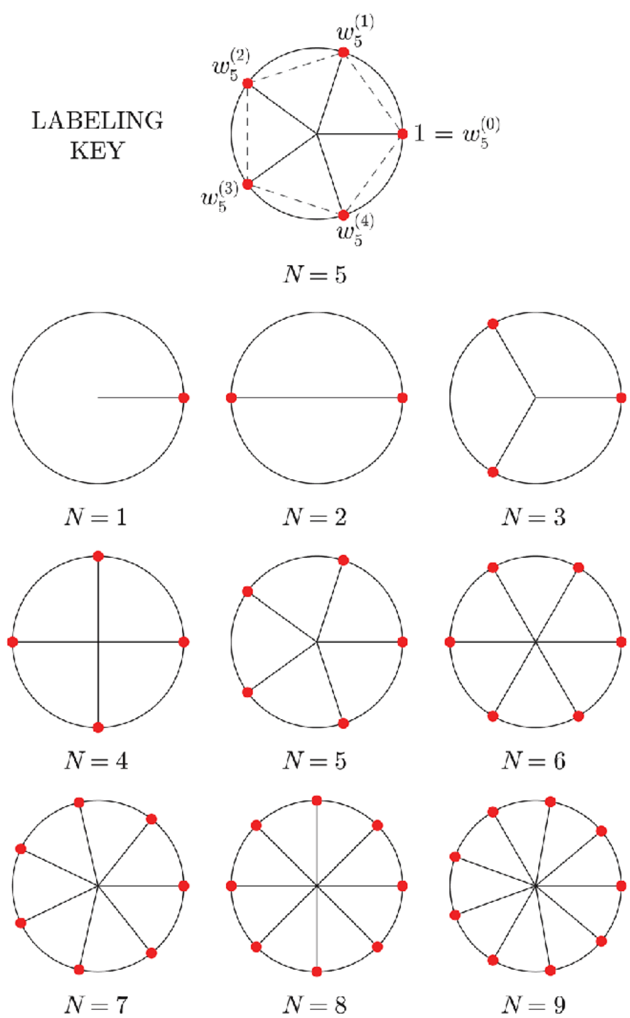
$$|z| = |a + ib| = \sqrt{a^2 + b^2}, \quad |z_1 \pm z_2| \leq |z_1| + |z_2|$$

Roots of Unity

Let $n > 1$ be a natural number. Solve the equation $z^n = 1$ and plot its solutions on the complex plane. To solve this, first observe that $z^n = 1$ means that $|z|^n = 1$, so that $|z| = 1$. This shows that the solutions will lie on the unit circle. Using the polar form,

$$z^n = \cos(n\theta) + i\sin(n\theta) \iff \cos n\theta = 1 \text{ and } \sin n\theta = 0 \iff n\theta = 2\pi k, \quad k \in \mathbb{Z}$$

The last expression gives us the θ for which the solutions exist, the appear as below in the plane.



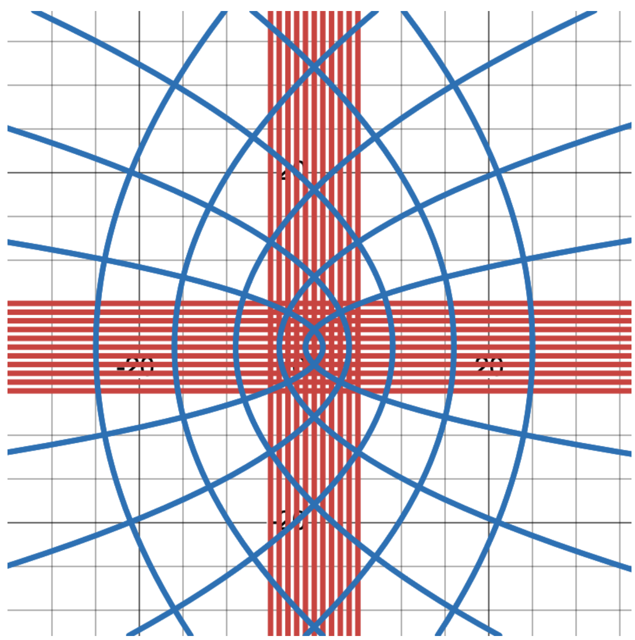
Connecting the vertices in the plane will form a regular n-gon.

1.2 Exploring Graphs of Complex Functions

Suppose we want to think about how the mapping $f : \mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^2$ transforms vertical and horizontal lines in the complex plane. If we consider $L = \{z \in \mathbb{C} : \mathrm{Re}(z) = 0\}$, then we see that $ib \mapsto -b^2$, which means that $f(L) = \{z \in \mathbb{C} : \mathrm{Re}(z) \leq 0, \mathrm{Im}(z) = 0\}$. If we take arbitrary vertical lines in the right half plane, that is $L = \{z \in \mathbb{C} : \mathrm{Re}(z) = a\}$, we see that elements in this set have the form $a + it$ for $t \in \mathbb{R}$ and

$$f(L) = \{a^2 - t^2 + i2at \mid t \in \mathbb{R}\}$$

It turns out that the image $f(L)$ will be a parabola if we consider its parametric formula. Set $x = a^2 - t^2$ and $y = 2at$. Then since $t = \frac{y}{2a}$, by substitution into the first equation, we have $x = a^2 - \frac{y^2}{(2a)^2}$. This is precisely a parabola that has a vertex at $(a^2, 0)$ and which opens to the left. To consider the images of horizontal lines, simply observe that we can consider $f(iL)$, where L is some vertical line. This shows us that horizontal lines map to parabolas whose vertexes are in the nonpositve x-axis and which open to the right. A picture is now given of this situation



We should note that parabolas corresponding to a family of vertical or horizontal lines are disjoint. To prove this, suppose that there exist z_1, z_2 on different vertical lines such that $f(z_1) = f(z_2)$. Observe that

$$z_1^2 = z_2^2 \iff (z_1 - z_2)(z_1 + z_2) = 0$$

which shows that $z_1 = z_2$, which cannot happen since z_1 and z_2 are distinct by assumption or $z_1 = -z_2$, which cannot happen as we have chosen z_1, z_2 to be on lines in the upper half

plane, meaning that their real parts are both greater than zero.
Further, the orthogonality of the points of intersection of these vertical and horizontal lines is preserved locally at the points to which these intersections are mapped.

1.3 Some Topology and Discussion On Holomorphicity

Definition. *Neighborhood*
Let $a \in \mathbb{C}$ and $r > 0$. Denote

$$D_{a,r} := \{z \in \mathbb{C} : |z - a| < r\}$$

This is just the open disc of radius r around a. Given some set $U \subseteq \mathbb{C}$, we call a an *interior point* if there exists some $\varepsilon > 0$ such that $D_{a,\varepsilon} \subseteq U$.

Definition. *Open*
A set $U \subseteq \mathbb{C}$ is open if every $a \in U$ is an interior point of U.

Examples.

1. A neighborhood is an open set.
2. $E = \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\}$ is not open.
3. $E = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ is open.

Definition. *Convergence*
Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers. The notation

$$\lim_{n\to\infty} z_n = w$$

has the usual meaning in the sense of convergence of sequences to some limit.

Note that $\lim_{n\to\infty} z_n = w$ if

$$\lim_{n\to\infty} |z_n - w| = 0$$

The definition of functional limit also follows the usual sense, that is the $\varepsilon - \delta$ definition.

Definition. *Continuity*
A function $f : U \to \mathbb{C}$ on an open subset $U \subseteq \mathbb{C}$ is continuous at $z_0 \in C$ if

$$\lim_{z\to z_0} f(z) = f(z_0), \quad \text{also} \lim_{z\in D} f(z) := \lim_{z\to z_0} f(z)$$

Definition. *Holomorphic*

Suppose $U \subseteq \mathbb{C}$ is an open subset and $f : U \to \mathbb{C}$ is a mapping. One says that f is holomorphic if for any $a \in U$, there exists a limit

$$f'(a) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This limit is alternatively written as

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

for $z_0 \in U$. One may also use the terms *complex differentiable* or *analytic* as synonyms to describe holomorphic functions.

Examples.

1. The function $f : \mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^2$ is holomorphic. This follows immediately from the definition
$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{(a+h)^2 - a^2}{h} = \lim_{h \to 0} 2a + h = 2a$$
2. Similarly, $z \mapsto z^n$ is holomorphic for $n \geq 1$, with $(z^n)' = nz^{n-1}$.
3. The function $f : \mathbb{C} \to \mathbb{C}$ given by $z \mapsto \bar{z}$ is not holomorphic. To show this, we can first consider the difference quotient
$$\frac{f(a+h) - f(a)}{h} = \frac{\overline{a+h} - \overline{a}}{h} = \frac{\overline{h}}{h}$$
If $h \in \mathbb{R}$, then $\lim_{h \to 0} \frac{\overline{h}}{h} = 1$, but if $h \in i\mathbb{R}$, then $\lim_{h \to 0} \frac{\overline{h}}{h} = -1$. Observe also that
$$\frac{\overline{h}}{h} = \frac{re^{-i\theta}}{re^{i\theta}} = e^{-2i\theta} = \cos(2\theta) - i\sin(2\theta)$$

Proposition. *Algebraic Differentiability.*

If f and g are holomorphic in Ω, then:

(i) $f + g$ is holomorphic in Ω and $(f+g)' = f' + g'$.

(ii) fg is holomorphic in Ω and $(fg)' = f'g + fg'$.

(iii) If $g(z_0) \neq 0$, then f/g is holomorphic at z_0 and

$$(f/g)' = \frac{f'g - fg'}{g^2}$$

Moreover, if $f : \Omega \to U$ and $g : U \to \mathbb{C}$ are holomorphic, the chain rule holds

$$(g \circ f)'(z) = g'(f(z))f'(z) \quad \text{for all } z \in \Omega$$

Corollary. Any polynomial in z (with complex coefficients), is holomorphic on $\mathbb{C}$. If P and Q are polynomials in $\mathbb{C}$, then $R(z) := \frac{P(z)}{Q(z)}$ is holomorphic at all points of $\mathbb{C}$, except at the roots of $Q(z)$.

1.4 The Exponential Function

We know that for $x \in \mathbb{R}$, we may write the Taylor series for e^x as

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Proposition. For any $z \in \mathbb{C}$, the series

$$1 + z + \frac{z^2}{2!} + \dots + \frac{z^n}{n!} + \dots$$

converges absolutely

Proof. Put $a := |z|$. The series of moduli

$$1 + a + \frac{a^2}{2!} + \dots + \frac{a^n}{n!} + \dots$$

converges by the ratio test. Observe that

$$\frac{a^{n+1}}{(n+1)!} \cdot \frac{n!}{a^n} = \frac{a}{n+1} \xrightarrow[n\to\infty]{} 0$$

■

Given the above proposition, it makes sense to make the following definition.

Definition. *Exponential function.*
For any $z \in \mathbb{C}$, put

$$e^z := \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

Our aim now is to show that e^z is holomorphic. We start with a fact.

Fact. Suppose $a \in \mathbb{C}$ and consider the series

$$\sum_{n=0}^{\infty} c_n(z-a)^n = c_0 + c_1(z-a) + c_2(z-a)^2 + \ldots + c_n(z-a)^n + \ldots \qquad (*)$$

If the series $(*)$ converges for $z = z_0$ and if $r < |z_0 - a|$, then the series $(*)$ converges uniformly on the $\overline{D}_{a,r} = \{z \in \mathbb{C} : |z-a| \leq r\}$.

This statement above is saying that if the power series converges at a point z_0, meaning $|z_0 - a|$ is within the radius of convergence, then if we choose any smaller radius $r > 0$, the series converges uniformly on the closed disk centered at a with radius r.

Proof. By the assumption, we know that $\sum_{n=0}^{\infty} c_n(z_0 - a)^n$ converges. Since the series converges, we must have $\lim_{n\to\infty} c_n(z_0 - a)^n = 0$. Since every convergent sequence is bounded, there exists an $M > 0$ such that $|c_n(z_0 - a)^n| \leq M$. Suppose $z \in \overline{D}_{a,r}$. We may write

$$\begin{aligned} |c_n(z-a)^n| = |c_n| \cdot |z-a|^n = |c_n||z_0 - a|^n \cdot \frac{|z-a|^n}{|z_0-a|^n} \\ \leq M \cdot \frac{|z-a|^n}{|z_0-a|^n} \\ \leq M \cdot \left(\frac{r}{|z_0-a|}\right)^n \end{aligned}$$

Now, since $\frac{r}{|z_0-a|} < 1$, it follows that the sequence consisting of terms $M \cdot \left(\frac{r}{|z_0-a|}\right)^n$ is a geometric series. The conclusion follows when we apply the Weierstrass M-test.

□

Aside. The Weierstrass M-test. If $\sum_{n=0}^{\infty} f_n(x)$ is a series of functions on X, and if there exists a convergent series $\sum b_n$ with $b_n \geq 0$ such that $|f_n(x)| \leq b_n$ for all $x \in X$, then $\sum_{n=0}^{\infty} f_n(x)$ converges uniformly on X.

Corollary. Suppose that $\sum_{n=0}^{\infty} c_n(z-a)^n$ is a power series. Then there exists an $R \in [0, +\infty]$ such that

1. $(*)$ converges uniformly on $\overline{D}_{a,r}$ whenever $r < R$.
2. $(*)$ diverges whenever $|z-a| > R$.

Remark. We remark, without proof, that

$$R = \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|c_n|}}$$

R is the radius of convergence of the series and this expression is called the Cauchy-Hadamard or Hadamard formula.

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Recall that

$$e^z = 1 + z + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots$$

converges for any $z \in \mathbb{C}$. We have shown previously that this means it converges absolutely and uniformly on any disk $\overline{D}_{0,R} = \{z : |z| \leq R\}$. Since uniform convergence preserves continuity, this implies that the right-hand side is a continuous function of z.

We turn to the main property of the exponential function

$$e^{z+w} = e^z \cdot e^w$$

Proof. Denote

$$e^z = 1 + z + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots$$

$$e^w = 1 + w + \frac{w^2}{2!} + \ldots + \frac{w^n}{n!} + \ldots$$

Since e^z and e^w are absolutely convergent, we see that

$$e^z \cdot e^w = 1 + (z + w) + \ldots + \sum_{k=0}^{n} \frac{z^{n-k}}{n-k!} \frac{w^k}{k!} + \ldots$$

Looking at the generic term, we see that

$$\frac{z^n}{n!} + \frac{z^{n-1}}{(n-1)!}w + \frac{z^{n-2}}{(n-2)!}\frac{w^2}{2!} + \ldots + \frac{w^n}{n!} = \frac{1}{n!}\left(z^n + \binom{n}{1}z^{n-1}w + \binom{n}{2}z^{n-2}w^2 + \ldots + w^n\right)$$

$$= \frac{(z+w)^n}{n!}$$

So

$$e^z \cdot e^w = 1 + (z + w) + \frac{(z+w)^2}{2!} + \ldots + \frac{(z+w)^n}{n!} + \ldots = e^{z+w}$$

□

Now, we justify that such a rearrangement is possible under the conditions.

Lemma. Suppose that $\sum a_n$ and $\sum b_n$ are **absolutely** convergent series of complex numbers, converging respectively to A and B. Put

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

for $n = 0, 1, 2, 3, \ldots$ Then the series $\sum c_n$ converges absolutely and

$$\sum_{n=0}^{\infty} c_n = AB$$

Proof of the Lemma. Put $A_n = \sum_{k=0}^n a_k$ and $B_n = \sum_{k=0}^n b_k$, and $C_n = \sum_{k=0}^n c_k$. Since $\lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} B_n$, it follows that $\lim_{n\to\infty} A_n B_n = AB$. So, it suffices to show that

$$\lim_{n\to\infty} (A_n B_n - C_n) = 0$$

Denote

$$A'_n = \sum_{k=0}^n |a_k|, \quad B'_n = \sum_{k=0}^n |b_k|$$

$$\lim_{n\to\infty} A'_n =: A', \quad \lim_{n\to\infty} B'_n =: B', \quad \lim_{n\to\infty} A'_n B'_n = A'B'$$

Consider the table of terms below.

a_0b_n	a_1b_n	a_2b_n	$\cdots$	a_nb_n
a_0b_1	a_1b_{n-1}	a_2b_{n-1}	$\cdots$	a_nb_{n-1}
a_0b_2	a_1b_{n-2}	a_2b_{n-2}	$\cdots$	a_nb_{n-2}
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
a_0b_0	a_1b_0	a_2b_0	$\cdots$	a_nb_0

Above, A_nB_n is the sum of all the entries in the table. C_n is the sum of the entries circled in red. $A_nB_n - C_n$ is the sum of the entries above the diagonal. By the triangle inequality, $|A_nB_n - C_n|$ is less than or equal to the sum of the moduli of the entries above the diagonal. It follows that

$$|A_nB_n - C_n| \le A'_nB'_n - A'_{\lfloor \frac{n}{2} \rfloor} B'_{\lfloor \frac{n}{2} \rfloor} \xrightarrow[n\to\infty]{} 0$$

The idea is that the value of the terms in the entries above the diagonal tend to zero as n increases.

□

2.1 The Derivative of the Exponential Function

Fix $a \in \mathbb{C}$. We claim that $(e^z)'_{z=a} = e^a$. Indeed

$$\frac{e^{a+h} - e^a}{h} = \frac{e^a \cdot e^h - e^a}{h} = e^a \cdot \frac{e^h - 1}{h}$$

So, if we manage to prove that $\lim_{h\to 0}\frac{e^h-1}{h}=1$, we will be done. Observe that

$$e^h-1=h+\frac{h^2}{2!}+\ldots+\frac{h^n}{n!}+\ldots$$
$$\frac{e^h-1}{h}=1+\frac{h}{2!}+\frac{h^2}{3!}+\ldots+\frac{h^n}{(n+1)!}+\ldots$$

if $h\neq 0$. Since we have only performed elementary operations on the series for e^h, which we know converges, it follows that the right-hand side converges for any $h\in\mathbb{C}$. Hence, this is a continuous function of h. Therefore, by continuity, the limit of right-hand side as $h\to 0$ is equal to 1 and this finishes the proof.

Conclusion. e^z is holomorphic on all $\mathbb{C}$ and $(e^z)'=e^z$.

2.2 The Euler Formula

Recall that $e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}$. Making the substitution $z=ix$, for $x\in\mathbb{R}$, we have

$$\begin{aligned} e^{ix}&=1+ix-\frac{x^2}{2!}-i\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots \\ &=\left(1-\frac{x}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\right)+i\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\right) \\ &=\cos x+i\sin x \end{aligned}$$

From now on, we will write $re^{i\varphi}$ instead of $r(\cos\varphi+i\sin\varphi)$. Taking $x\in\mathbb{R}$ still, notice that

$$e^{ix}=\cos x+i\sin x,\qquad e^{-ix}=\cos x-i\sin x$$

This implies

$$\cos x=\frac{e^{ix}+e^{-ix}}{2},\qquad \sin x=\frac{e^{ix}-e^{-ix}}{2i}$$

Thus, we define the complex trigonometric functions of sine and cosine in an analogous manner, replacing the argument $x\in\mathbb{R}$ with $z\in\mathbb{C}$. One may verify that the usual derivatives of sine and cosine hold with this definition.

Further, the chain rule also holds.

Chain rule. Suppose $U,V\subseteq\mathbb{C}$ are open subsets and $f:U\to V$ and $g:V\to\mathbb{C}$ are holomorphic. Then $g\circ f$ is holomorphic and if $a\in U$, then

$$(g\circ f)'(a)=g'(f(a))\cdot f'(a)$$

Finally, the usually trigonometric identities, and identities subsequently derived, also hold for all $z,w\in\mathbb{C}$.

$$\sin(z+w)=\sin z\cos w+\cos z\sin w$$
$$\cos(z+w)=\cos z\cos w-\sin z\sin w$$
$$\cos^2(z)+\sin^2(z)=1$$

2.3 Integrals

We begin with some definitions.

1. **Paths.** A piecewise smooth path in the complex plane is a piecewise smooth mapping $\gamma : [p, q] \to \mathbb{C}$, where $[p, q] \in \mathbb{R}$. Letting γ be given by $t \mapsto u(t) + iv(t)$, this means that u and v should be piecewise C^1-smooth. If not specified, all paths will be considered piecewise smooth.

2. We may denote the image as
$$\gamma([p, q]) =: |\gamma| \subseteq \mathbb{C}$$

Definition. *Line Integral*
Suppose that $f : |\gamma| \to \mathbb{C}$ is continuous.
$$\int_\gamma f(z)\, dz := \int_p^q f(\gamma(t))\gamma'(t)\, dt$$
where
$$\gamma'(t) = u'(t) + iv'(t) = \lim_{h\to 0} \frac{\gamma(t+h) - \gamma(t)}{h}$$

Informally, one might say
$$z = \gamma(t), \quad dz = \gamma'(t)\, dt$$
as t varies from p to q, thinking of using a u-substitution in the integral above. Note that the definition depends on the parametrization (though it is *almost* independent).

Proposition. Suppose we have
$$[p_1, q_1] \xrightarrow{\varphi} [p, q] \xrightarrow{\gamma} \mathbb{C}$$
Suppose that γ is a piecewise smooth path and φ is a smooth increasing bijection. Then
$$\int_\gamma f(z)\, dz = \int_{\gamma\circ\phi} f(z)\, dz$$

Proof. By direction computation, we have
$$\begin{aligned}\int_\gamma f(z)\, dz = \int_p^q f(\gamma(t))\gamma'(t)\, dt &= \int_{p_1}^{q_1} f(\gamma(\varphi(s))\gamma'(\varphi(s))\, \varphi'(s)ds \\ &= \int_{p_1}^{q_1} f((\gamma\circ\varphi)(s))(\gamma\circ\varphi)'(s)\, ds \\ &= \int_{\gamma\circ\varphi} f(z)\, dz\end{aligned}$$

Note that this almost shows independence of the integral from parametrization: if the bijection is monotonically decreasing rather than increasing, then the integral will have a negative sign in front. So, it turns out the dependence is related to the orientation of the path.

□

Example. Let $|\gamma| = \{z : |z| = 1\}$ Find $\int_\gamma dz$.
Solution. Consider the path $z = \gamma(t) = \cos t + i\sin t$ for $t \in [0, 2\pi]$. Then $dz = -\sin t + i\cos t$. Computing, we have

$$\begin{aligned}\int_\gamma dz &= \int_\gamma \gamma'(t)\,dt \\ &= \cos t + i\sin t\Big|_0^{2\pi} = 0\end{aligned}$$

Alternatively

$$\int_\gamma dz = \int_0^{2\pi} ie^{it}\,dt = e^{it}\Big|_0^{2\pi} = 0$$

Proposition. *Newton-Leibniz formula in the complex plane.* Suppose that $U \subseteq \mathbb{C}$ is open and $F : U \to \mathbb{C}$ is holomorphic (this implies that $F' : U \to \mathbb{C}$ is exists and is continuous). Let $\gamma : [p, q] \to U$ be a path in U that joins the points a and b. Then

$$\int_\gamma F'(z)\,dz = F(b) - F(a)$$

Proof. By direct computation

$$\begin{aligned}\int_\gamma F'(z)\,dz &= \int_p^q F'(\gamma(t))\gamma'(t)\,dt = \int_p^q (F\circ\gamma)'(t)\,dt \\ &= F(\gamma(t))\big|_p^q = F(\gamma(q)) - F(\gamma(p)) = F(b) - F(a)\end{aligned}$$

■

Note that the Newton-Leibniz formula shows that integrating over any closed paths gives zero.
Example 2. Let $|\gamma| = \{z : |z| = 1\}$, with endpoints at 1, let counterclockwise be the positive orientation, and let $n \in \mathbb{Z}$. Find $\int_\gamma z^n\,dz$.
Solution. Suppose that $n \neq -1$. Then we have

$$z^n = \left(\frac{z^{n+1}}{n+1}\right)'$$

Then, by the Newton-Leibniz formula, we see that $\int_\gamma z^n\,dz = 0$. On the other hand, if $n = -1$, then we have

$$\int_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\,dt = \int_0^{2\pi} i\,dt = 2\pi i$$

Also, consider $\gamma_{a,r} = \{z : |z - a| = r\}$. This has a parametrization of $\gamma = a + re^{it}$, with $t \in [0, 2\pi]$. So with this path

$$\int_{\gamma_{a,r}} \frac{dz}{z-a} = \int_0^{2\pi} \frac{rie^{it}}{re^{it}} = \int_0^{2\pi} i\,dt = 2\pi i$$

Above, we have our first instance of a fact. The integral of a holomorphic function over as closed path does not change between paths that are continuous deformations of each other. So, given that γ_1 and γ_2 are homotopic, we have

$$\int_{\gamma_1} \frac{dz}{z} = \int_{\gamma_2} \frac{dz}{z}$$

The Line Integral is Linear.

$$\int_\gamma (af(z) + bg(z))\,dz = a\int_\gamma f(z)\,dz + b\int_\gamma g(z)\,dz$$

2.4 Lengths of Paths

Let $\gamma : [p, q] \to \mathbb{C}$ be a path. We define the length ℓ of γ as

$$\ell(\gamma) := \int_p^q |\gamma'(t)|\,dt$$

Proposition.

$$\left|\int_\gamma f(z)\,dz\right| \leq \sup_{z\in|\gamma|} |f(z)| \cdot \ell(\gamma)$$

Proof. Let $M = \sup_{z\in|\gamma|} |f(z)|$. Observe that

$$\begin{aligned}\left|\int_\gamma f(z)\,dz\right| &= \left|\int_\gamma f(\gamma(t))\gamma'(t)\,dt\right| \\ &\leq \int_\gamma |f(\gamma(t))| \cdot |\gamma'(t)|\,dt \\ &\leq \int M \cdot |\gamma'(t)|\,dt = M \cdot \ell(\gamma)\end{aligned}$$

where we have used the inequality

$$\left|\int_p^q \varphi(t)\,dt\right| \leq \int_p^q |\varphi(t)|\,dt$$

and where φ is complex-valued.

□

A less trivial estimate. Let $\gamma : [p,q] \to \mathbb{C}$ be a path and let $f : |\gamma| \to \mathbb{C}$ be a continuous function.

$$\int_\gamma f(z)\,|dz| := \int_p^q f(\gamma(t)) \cdot |\gamma'(t)|\,dt$$

Example. By the above definition, we have

$$\int_\gamma |dz| = \ell(\gamma)$$

Moving on, we have the estimate

$$\left|\int_\gamma f(z)\,dz\right| \leq \int_\gamma |f(z)| \cdot |dz|$$

Proof. Observe that

$$\begin{aligned}
\left|\int_\gamma f(z)\,dz\right| &= \left|\int_\gamma f(\gamma(t))\gamma'(t)\,dt\right| \\
&\leq \int_p^q |f(\gamma(t))| \cdot |\gamma'(t)|\,dt \\
&= \int_\gamma |f(z)| \cdot |dz|
\end{aligned}$$

□

3 September 27, 2024

3.1 Chain Rule

We begin by recalling and proving the chain rule.

Chain rule. Suppose $U, V \subseteq \mathbb{C}$ are open subsets and $g : U \to V$ and $f : V \to \mathbb{C}$ are holomorphic. Then $f \circ g$ is holomorphic and if $a \in U$, then

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$$

Proof. Since g is holomorphic, $g'(a) = M$ exists. As a result, we may write

$$g(a+h) = g(a) + Mh + \varphi(h)$$

where $\lim_{h\to 0} \frac{|\varphi(h)|}{|h|} = 0$. This is saying that the numerator shrinks faster than the denominator, in other words, $\varphi(h) = o(|h|)$ as $h \to 0$. We may write

$$\frac{\varphi(h)}{h} = \frac{g(a+h) - g(a)}{h} - M$$

In the same vein, we have

$$\begin{aligned}
g(a+h) &= g(a) + g'(a)h + \varphi(h), \quad \varphi(h) = o(|h|) \\
f(g(a)+k) &= f(g(a)) + f'(g(a)) \cdot k + \psi(k), \quad \psi(k) = o(|k|) \\
f(g(a+h)) &= f(g(a) + \underbrace{g'(a) \cdot h + \varphi(h)}_{k}) \\
&= f(g(a)) + f'(g(a)) \cdot (g'(a) \cdot h + \varphi(h)) + \psi(g'(a) \cdot h + \varphi(h)) \\
&= f(g(a)) + f'(g(a)) \cdot g'(a) \cdot h \\
&\quad + \underbrace{f'(g(a)) \cdot \varphi(h) + \psi(g'(a) \cdot h + \varphi(h))}_{\text{error term}}
\end{aligned}$$

It suffices to show that each summand is $o(|h|)$. For the first summand, we see that

$$\frac{f'(g(a)) \cdot \varphi(h)}{h} = f'(g(a)) \cdot \frac{\varphi(h)}{h} \xrightarrow[h \to 0]{} 0$$

For the second summand, we know that $\left|\frac{\psi(k)}{k}\right| \xrightarrow[k\to 0]{} 0$. Put

$$\omega(t) = \sup_{0 \le |k| \le t} \frac{\psi(k)}{k}$$

This shows that there exists a function $\omega : (0, \varepsilon) \to \mathbb{R}^+$ such that

$$|\psi(k)| \le \omega(|k|) \cdot |k|$$

and $\lim_{t\to 0}\omega(t) = 0$. Now, we compute

$$\begin{aligned}|\psi(g'(a)\cdot h + \varphi(h))| &\leq \omega(|g'(a)\cdot h + \varphi(h)|)\cdot |g'(a)\cdot h + \varphi(h)|\\ &\leq \omega(|g'(a)\cdot h + \varphi(h)|)\cdot |g'(a)\cdot h| + \omega(|g'(a)\cdot h + \varphi(h)|)\cdot |\varphi(h)|\end{aligned}$$

Dividing through by $|h|$ on both sides, we see that

$$\frac{|\psi(g'(a)\cdot h + \varphi(h))|}{|h|} \leq \omega(\ldots)\cdot |g'(a)| + \omega(\ldots)\cdot \frac{|\varphi(h)|}{|h|}$$

Notice that the right hand side approaches zero as $h \to 0$, and the conclusion follows.

□

3.2 Inverse Function Theorem and Holomorphicity

Theorem. *Inverse Function Theorem.*
Suppose that $U, V \subseteq \mathbb{C}$ are open sets and

1. $f : U \to V$ is a holomorphic bijection
2. $f'(a) \neq 0$ for any $a \in U$
3. $f^{-1} : V \to U$ is continuous

Then, $f^{-1} : V \to U$ is also holomorphic. If $a \in U$ and $f(a) = b$, then

$$(f^{-1})'(b) = \frac{1}{f'(a)}$$

holds.

Letting $g = f^{-1}$, we can derive the above relation by considering that $(g \circ f)(x) = x$ and then taking the derivative and using the chain rule.
Remark. With a holomorphic bijection, conditions 2 and 3 are actually redundant.
Exercise. Let $H = \{z : \operatorname{Im}(z) > 0\}$ be the open upper half plane and consider the mapping $f : z \mapsto z^2$. What is $f(H)$?
Solution. We can rewrite the set H as $H = \{re^{i\varphi} : r > 0,\ 0 < \varphi < \pi\}$. Then, using the fact that multiplication of complex numbers corresponds to multiplying the moduli and adding the angles, we see that

$$f(H) = \{\rho e^{i\psi} : \rho > 0,\ 0 < \varphi < 2\pi\} = \mathbb{C} \setminus [0, \infty)$$

Follow up. We will denote $V := f(H)$. Is $f : H \to V$ a bijection?
Since the codomain is the image of f, we see that f is surjective by definition. It remains to check injectivity. We can proceed in two ways. First, we can let $h_1, h_2 \in H$ and assume

$h_1 \neq h_2$. Considering $r_1 e^{2i\varphi_1} = h_1^2 = h_2^2 = r_2 e^{2i\varphi_2}$. If $r_1 \neq r_2$, then we are done. If $r_1 = r_2$, then we must have $\varphi_1 - \varphi_2 = \pi$. But this is not possible since $\varphi_1, \varphi_2 \in (0, \pi)$, so $h_1^2 \neq h_2^2$ and the function is injective.

A second method is to consider $z_1, z_2 \in H$ and assume $z_1^2 = z_2^2$. Then we must have $(z_1 - z_2)(z_1 + z_2) = 0$. But since both points lie in H, we cannot have $z_1 = -z_2$, so $z_1 = z_2$ and we are done.

Recall that we have shown previously that $f : H \to V$ given by $z \mapsto z^2$ is holomorphic, with $f'(z) = 2z \neq 0$, if $z \in H$. Since f is a holomorphic bijection, it is also a homeomorphism.

Example. Consider a new function $g : V \to H$ given by $z \mapsto \sqrt{z}$. We know by definition that $\sqrt{-1} = i$. Suppose we want to calculate $g'(-1)$. We can consider $f(a) = b$ so that $f(i) = -1$; here, f still means squaring. Then, making substitutions, we have

$$g'(b) = \frac{1}{f'(a)} \implies g'(-1) = \frac{1}{f'(i)} = \frac{1}{2i} = -\frac{i}{2}$$

Remark. One cannot define a holomorphic function $z \mapsto \sqrt{z}$ on the entire $\mathbb{C}$!

Proposition. There is no continuous function $\arg : \mathbb{C} \setminus \{0\} \to \mathbb{R}$.

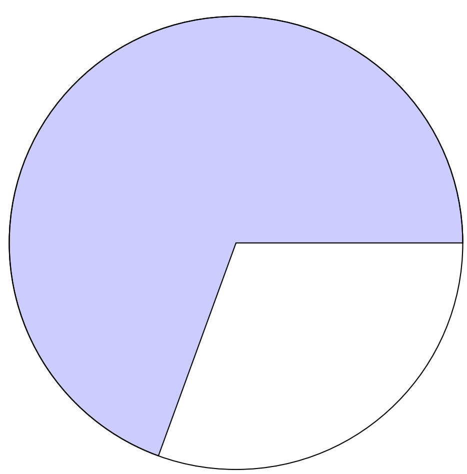
Proof Sketch. Consider points a, b in the fourth quadrant of the complex plane which are distinct, but very close to the positive x-axis. Let $\varepsilon_1, \varepsilon_2 > 0$ and let both ε_1 and ε_2 be very small. Then, we have

$$\arg(\sqrt{a}) = \pi - \varepsilon_1, \quad \arg(\sqrt{b}) = \pi - \varepsilon_2$$
$$\arg(\sqrt{a} \cdot \sqrt{b}) = 2\pi - (\varepsilon_1 + \varepsilon_2)$$

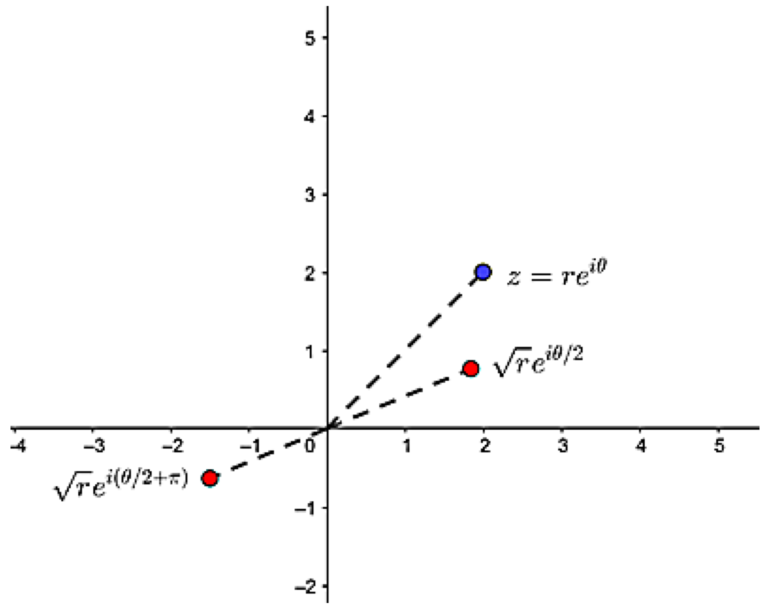
Notice how $\arg(\sqrt{a} \cdot \sqrt{b}) \notin (0, \pi)$. Thus, this value cannot be equal to the argument of $\sqrt{ab}$, with our choice of the branch of square root as H (see how g was defined above).

□

Remark. Actually, $\sqrt{\cdot}$ exists on any sector.



Letting $z = \sqrt{re^{i\varphi}} = \sqrt{r}e^{i\frac{\varphi}{2}}$, then φ is a continuous function of z on this shaded blue sector above. Recall that with the real square root function $f(x) = \sqrt{x}$, there is a choice of a positive root and a negative root, which we can specify easily. However, given an arbitrary complex number $z = re^{i\varphi}$, we also have two choices for the square root, $\sqrt{r}e^{i\frac{\varphi}{2}}$ and $\sqrt{r}e^{i\left(\frac{\varphi}{2}+\pi\right)}$. A diagram describing the situation is given below



These two possible expressions for the result of the complex square root are called the *branches* of the square root. We can specify a branch by stating $\sqrt{-1} = i$ or $\sqrt{-1} = -i$.

3.3 Logarithms

We begin by considering the upper half plane again $H = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$. Let $f : H \to \mathbb{C}$ be given by $z \mapsto e^z$. Notice that we have

$$e^z = e^x e^{iy} = e^x(\cos(y) + i\sin(y))$$

So, the image is $f(H) = \mathbb{C} \setminus \{0\}$. Further, the exponential function on H is not bijective since $e^{z+2\pi i} = e^z$. To make the function bijective, we consider a restriction to the strip of the upper half plane

$$U = \{z \in \mathbb{C} : 0 < \text{Im}\, z < 2\pi\}$$

The image will then be $f(U) = \{e^x e^{iy} : x > 0, 0 < y < 2\pi\} = \mathbb{C} \setminus [0, \infty) =: V$. In order to define the inverse function, the logarithm, we want $g : V \to U$ with $z \mapsto \log(z)$ and we require $0 < \text{Im}(z) < 2\pi$. In the polar form, we have

$$re^{i\varphi} \mapsto \log r + i\varphi$$

Notice that $\log(r)$ will be uniquely determined and one should choose φ continuously.

3.4 Cauchy-Riemann Equations

Let $U \subseteq \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic. Denote $f(x+iy) = u(x,y) + iv(x,y)$. If f is holomorphic, then what restrictions does it impose on u and v?

Theorem.
f is holomorphic if and only if it satisfies the Cauchy-Riemann equations.

Proposition. Suppose that f is holomorphic and $c \in U$. Then we have

$$\begin{pmatrix} \frac{\partial u}{\partial x}(c) & \frac{\partial u}{\partial y}(c) \\ \frac{\partial v}{\partial x}(c) & \frac{\partial v}{\partial y}(c) \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

for some $a, b \in \mathbb{R}$. In other words, we have the relations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The system above represents the **Cauchy-Riemann** equations.

Proof. Let $f(x+iy) = u(x,y) + iv(x,y)$ and let $f'(c) = a + bi$, where $c = x + iy$. Suppose $h \in \mathbb{R}$ and note that $c + h = (x+h) + iy$. Then

$$\begin{aligned} \frac{f(c+h) - f(c)}{h} &= \frac{u(x+h,y) + iv(x+h,y) - u(x,y) - iv(x,y)}{h} \\ &= \frac{u(x+h,y) - u(x,y)}{h} + i \cdot \frac{v(x+h,y) - v(x,y)}{h} \\ &\xrightarrow[h \to 0]{} \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = a + bi = f'(c) \end{aligned}$$

Now, with $h = it$ and $t \in \mathbb{R}$, a similar argument shows that $a + bi = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$. The order of the partial derivatives switches because of the term it in the denominator. The conclusion follows by comparing values of a and b.

□

Proposition. Suppose that $U \subseteq \mathbb{C}$ is open and $u, v : U \to \mathbb{R}$ are C^1-functions and at any point of U

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Then the function $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic.

Proof. Since its components are C^1, it follows that f is also C^1. We may compute

$$f(x+h, y+k) = f(x,y) + M\begin{pmatrix} h \\ k \end{pmatrix} + o(\sqrt{h^2+k^2})$$

$$M\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} ah - bk \\ bh + ak \end{pmatrix}$$

$$= (ah - bk) + i(bh + ak) = (a+bi)(h+ik)$$

Let $s = h + ik$. We have

$$f(z+s) = f(z) + (a+bi)s + o(|s|)$$

So, $f'(z)$ exists and it is equal to $a + bi$.

$\square$

Remark. The matrices of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

are matrices of linear operators $\mathbb{C} \to \mathbb{C}$ with $z \mapsto \lambda z$ where $\lambda = a + bi$ in the basis $\{1, i\}$. Recall from vector calculus, for a function $f : \mathbb{R}^2 \to \mathbb{R}^2$, the derivative is a 2×2 Jacobian matrix. The Cauchy-Riemann equations give an additional restriction on the entries of this matrix.

3.5 Preservation of Angles

Proposition. Suppose $U \subseteq \mathbb{C}$ is open and $f : U \to \mathbb{C}$ is holomorphic and $f'(a) \neq 0$ at some $a \in \mathbb{C}$. Let γ_1 and γ_2 be smooth curves passing through a. Then, the angle between γ_1 and γ_2 at a will be the same as the angle between $f(\gamma_1)$ and $f(\gamma_2)$ at $f(a)$.

Proof. Without loss of generality, let $\gamma_1, \gamma_2 : (-\varepsilon, \varepsilon) \to \mathbb{C}$ with $\gamma_1(0) = \gamma_2(0) = a$ and $\gamma_1'(0) \neq 0$ and $\gamma_2'(0) \neq 0$ (the curves are smooth). Let $f'(a) = k$. Considering the parametrizations of $f \circ \gamma_1$ and $f \circ \gamma_2$, we have

$$\left.\frac{d}{dt} f(\gamma_1(t))\right|_{t=0} = \underbrace{f'(\gamma_1(0))}_{k} \cdot \gamma_1'(0)$$

$$\left.\frac{d}{dt} f(\gamma_2(t))\right|_{t=0} = \underbrace{f'(\gamma_2(0))}_{k} \cdot \gamma_2'(0)$$

This shows that $\gamma_1'(0)$ and $\gamma_2'(0)$, these are the tangent vectors to the curve, are rotated by the same amount. Therefore, their angles are preserved.

$\square$

Remark. At the points where $f' = 0$, the angles are **not** preserved!
Example. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $z \mapsto z^2$ and consider $a = 0$ with rays for γ_1, γ_2.

4 October 4, 2024

Our goal today will be to integrate complex functions over closed paths.

4.1 Indices of Curves

Let $\gamma : [p, q] \to \mathbb{C}$ be a closed curve, that is, it satisfies $\gamma(p) = \gamma(q)$. Assume $a \notin |\gamma|$. An informal definition of the index of a curve γ at a point a is the number of times that γ winds around a, so we have $\text{ind}_a(\gamma) \in \mathbb{Z}$.

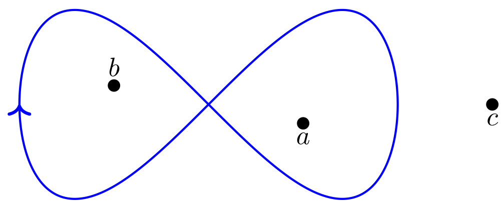
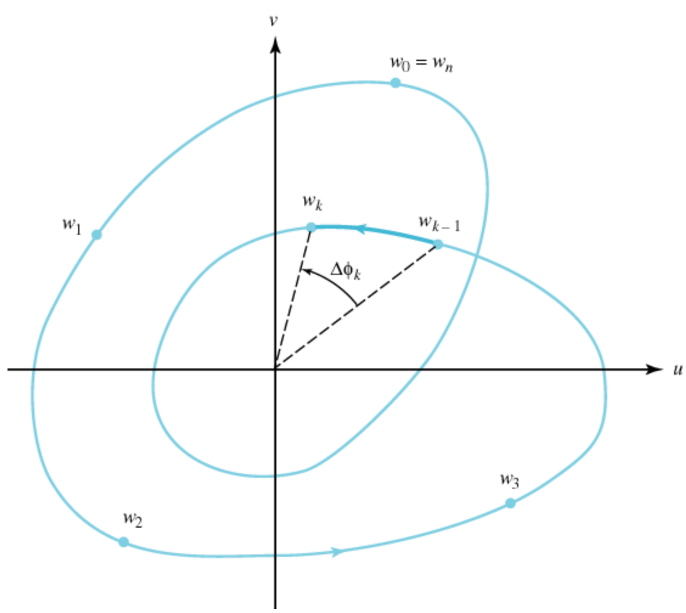


Figure 1: Informal Notion of Index

If we consider the counterclockwise direction positive and the clockwise direction positive, we have

$$\text{ind}_a(\gamma) = 1, \quad \text{ind}_b(\gamma) = -1, \quad \text{ind}_c(\gamma) = 0$$

A less informal definition involves explicitly keeping track of the total angle through which we have rotated as we trace along the curve, relative to the point. This is shown below.



Here, we define $\text{ind}_a \gamma$ as $\frac{1}{2\pi}$ multiplied by the increment of $\arg(z - a)$ as z travels once along the path. Generally, $\varphi_k = \arg(z_k - a) - \arg(z_{k-1} - a) > 0$. We now discuss the precise definition.

Lemma. Let $\gamma : [p, q] \to \mathbb{C}$ be a closed curve, that is $\gamma(p) = \gamma(q)$ and take $a \notin |\gamma|$. There exists a continuous function $r : [p, q] \to \mathbb{R}_+ := (0, +\infty)$ and a continuous function $\varphi : [p, q] \to \mathbb{R}$ such that $\gamma(t) - a = r(t)e^{i\varphi(t)}$.

Proof. We may simply take $r(t) = |\gamma(t) - a|$ as our continuous function. To prove the existence of φ, let $S_1, \ldots, S_n$ be open sectors (around a) such that $\mathbb{C} \setminus \{a\} = \bigcup_{i=1}^{n} S_i$. Observe that the sectors may overlap.

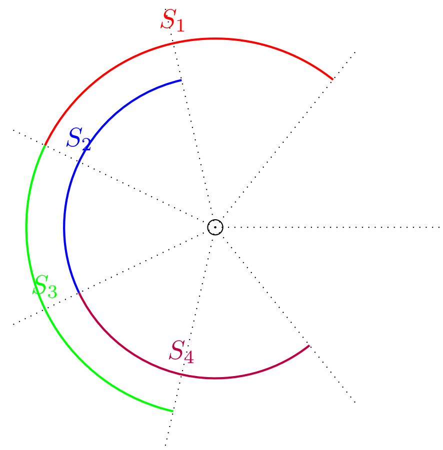


Figure 2: The Sectors S_i

Since $\gamma : [p, q] \to \mathbb{C} \setminus \{a\}$ is continuous, it follows that

$$[p, q] = \gamma^{-1}(S_1) \cup \ldots \cup \gamma^{-1}(S_n)$$

where each $\gamma^{-1}(S_i)$ is open.

Fact. *Lebesgue's number lemma.* If $[p, q] \subseteq \mathbb{R}$ and if $[p, q] = U_1 \cup \ldots \cup U_n$ (where U_i are open subsets), then there exists a subdivision

$$p = t_0 < t_1 < \ldots < t_{m-1} < t_m = q$$

such that $[t_j, t_{j+1}]$ is contained in some U_i for each j.

By this fact, we know that such a subdivision for $[p, q]$ exists and that $[t_j, t_{j+1}] \subseteq \gamma^{-1}(S_i)$ for some i. We now construct φ in steps.

Step 0. From the previous discussion, we have

$$\gamma([t_0, t_1]) \subseteq S_i$$

for some $i \in \{1, \ldots, n\}$. There exists a continuous function $S_i \to \mathbb{R}$ given by the mapping $z \mapsto \arg(z - a)$. We indicate $\arg_0 := \arg$. We define φ on the interval

$p = t_0 \leq t \leq t_1$ by

$$\varphi(t) = \arg_0(\gamma(t) - a)$$

Step 1. We have $\gamma([t_1, t_2]) \subseteq S_j$ for some $j \in \{1, \ldots, n\}$. Similar to the previous step, there exists a continuous function $z \mapsto \arg(z - a)$. Let $\arg_1 := \arg$. We would like to extend $\varphi(t)$ to $[t_1, t_2]$, but since we have already defined $\varphi(t)$ on $[t_0, t_1]$, the extension must agree with what we have already defined. Notice that

$$\arg_1(\gamma(t_1) - a) - \arg_0(\gamma(t_1) - a) = 2\pi N, \quad N \in \mathbb{Z}$$

Hence, we define the extension as

$$\varphi(t) = \arg_1(\gamma(t) - a) - 2\pi N$$

for $t \in [t_1, t_2]$.

In a finite number of steps, we will reach $q = t_m$, and through this construction we will obtain $\varphi : [p, q] \to \mathbb{R}$.

□

Definition. *Index*
With the lemma above, we can now give the precise definition of the index of a curve as:

$$\operatorname{ind}_a(\gamma) := \frac{\varphi(q) - \varphi(p)}{2\pi}$$

Remark. In the construction above, if γ is piecewise smooth, then r and φ may be chosen piecewise smooth.

Proposition. Suppose that $\gamma : [p, q] \to \mathbb{C} \setminus \{a\}$ is a closed and piecewise smooth path in $\mathbb{C}$, $\gamma(t) = r(t)e^{i\varphi(t)} + a$, $r(t) > 0$ and $\varphi(t) \in \mathbb{R}$. Then

$$\int_\gamma \frac{dz}{z - a} = i(\varphi(q) - \varphi(p))$$

Proof. Let $z = a + r(t)e^{i\varphi(t)}$, then $dz = r'(t)e^{i\varphi(t)} + r(t)ie^{i\varphi(t)} \cdot \varphi'(t)\, dt$. Substituting, we have

$$\begin{aligned}
\int_\gamma \frac{dz}{z - a} &= \int_p^q \left(\frac{r'(t)}{r(t)} + i\varphi'(t) \right) dt \\
&= \int_p^q \frac{r'(t)}{r(t)}\, dt + i \int_p^q \varphi'(t)\, dt \\
&= \log(r(t)) \Big|_p^q + i\varphi(t) \Big|_p^q = i(\varphi(q) - \varphi(p))
\end{aligned}$$

□

Corollary. $\operatorname{ind}_a \gamma$ is well-defined for piecewise smooth γ. (Actually, it is well-defined for arbitrary continuous γ.)

Proposition. If γ is a piecewise smooth closed path in the complex plane, and if $a \notin |\gamma|$, then

$$\operatorname{ind}_a \gamma = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}$$

Remark. If we deform γ continuously so that it never passes through a, then $\operatorname{ind}_a \gamma$ remains constant. Hence $\int_\gamma \frac{dz}{z-a}$ also remains constant. It is a particular case of the following fact.

Fact. If $U \subseteq \mathbb{C}$ is open, $f : U \to \mathbb{C}$ is holomorphic and γ is a closed path in U, then $\int_\gamma f(z)\, dz$ does not change as one deforms γ continuously in the class of closed paths in U.

In our case, we have $U = \mathbb{C} \setminus \{a\}$ and $f(z) = \frac{1}{z-a}$.
Remark. Regarding the fact, if we take $f(x, y) = u(x, y) + iv(x, y)$, and $z = x + iy$ with the differential $dz = dx + i\, dy$, it follows that

$$f(z)\, dz = (u + iv)(dx + i\, dy) = u\, dx - v\, dy + i(u\, dy + v\, dx)$$

Since f is holomorphic, the Cauchy-Riemann equations show that both the real and imaginary parts of the expression have no curl. In the real part, we have $-v_x - u_y = 0$ and in the imaginary part, we have $u_x - v_y = 0$.
Moving on, we explore some of the properties of $\operatorname{ind}_a \gamma$. First, notice that $\mathbb{C} \setminus |\gamma|$ is a union of disjoint, connected open subsets. We can see that this complement is a disjoint union by considering the equivalence classes of points that are path-connected.

Proposition. The function $a \mapsto \operatorname{ind}_a \gamma$ is continuous on $\mathbb{C} \setminus |\gamma|$.

Proof. It suffices to show $a \mapsto \int_\gamma \frac{dz}{z-a}$ is continuous on $\mathbb{C} \setminus |\gamma|$. To that end, suppose $a_n \to a$; we will show that

$$\int_\gamma \frac{dz}{z-a_n} \to \int_\gamma \frac{dz}{z-a}$$

Observe that

$$\begin{aligned}\left| \int_\gamma \frac{dz}{z-a_n} - \int_\gamma \frac{dz}{z-a} \right| &= \left| \int_\gamma \left(\frac{1}{z-a_n} - \frac{1}{z-a} \right) \right| dz \\ &\le \ell(\gamma) \cdot \sup_{z \in |\gamma|} \left| \frac{1}{z-a_n} - \frac{1}{z-a} \right|\end{aligned}$$

We may write

$$\left|\frac{1}{z-a_n} - \frac{1}{z-a}\right| = \left|\frac{a_n - a}{(z-a_n)(z-a)}\right|$$

We may choose a disk D so that $|b-z| \geq c > 0$ for $b \in D$, $z \in |\gamma|$. With respect to our expression, b will take the place of a and a_n, for sufficiently large n. To give more details, since $a_n \to a$, there exists a neighborhood around a that contains all points of the sequence after some $N \in \mathbb{N}$. To show that such a c exists, we can simply take

$$c = \inf\{|b-z| : b \in D\}$$

We give a visual idea below, keep in mind that $\mathbb{C} \setminus |\gamma|$ may be a much more complicated path with self intersections.

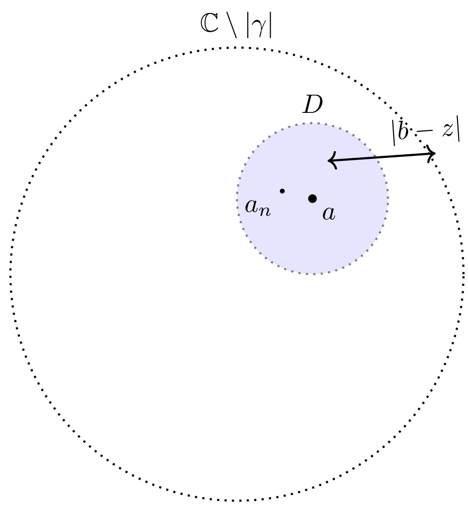


Figure 3: Choosing D and c

Then

$$\left|\frac{a_n - a}{(z-a_n)(z-a)}\right| \leq |a_n - a| \cdot \frac{1}{c^2} \to 0$$

□

4.2 Linear Fractional Mappings

Definition. *Linear Fractional Mapping*
Let $a, b, c, d \in \mathbb{C}$. A linear fractional mapping is defined by

$$z \mapsto \frac{az+b}{cz+d}, \quad \text{where} \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$$

Example. Let $L = \{z : \text{Re}(z) = 1\}$ and $f(z) = \frac{1}{z}$. What is $f(L)$?

One way to approach this problem is to consider

$$f(z) = \frac{1}{z} =: w$$

Then $w = f(z)$, so $z = \frac{1}{w} \in L$. Let $w = x + iy$ for $x, y \in \mathbb{R}$. We may compute

$$\frac{1}{w} = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x}{x^2+y^2} - i(\dots)$$

Since $\frac{1}{w} \in L$, we know that $\frac{x}{x^2+y^2} = 1$. We can put this equation into the form of a circle:

$$\left(x - \frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2$$

Since $w = x + iy$, this means that $f(L)$ is a circle of radius $\frac{1}{2}$ at center $\left(\frac{1}{2}, 0\right)$.

4.2.1 The Riemann Sphere

- Let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The Riemann sphere consists of the complex plane and a point at infinity.
- If $\lim_{n\to\infty} |z_n| = \infty$, one says that $\lim_{n\to\infty} z_n = \infty$.
- One regards linear fractional transformations as mappings $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$.

Given a linear fractional mapping $f(z) = \frac{az+b}{cz+d}$, how do we determine $f(\infty)$?
Observe that

$$\frac{az+b}{cz+d} = \frac{a + \frac{b}{z}}{c + \frac{d}{z}} \xrightarrow[|z|\to\infty]{} \frac{a}{c}$$

Hence $f(\infty) = \frac{a}{c}$. Similarly, if $z = -\frac{d}{c}$, then $f(z) = \infty$.

Definition. *Generalized Circle*
A generalized circle is either a circle or a line plus the point at infinity.

Proposition. Linear fractional mappings map generalized circles to generalized circles.

Proof. We may write

$$f(z) = \frac{az+b}{cz+d} = A + \frac{q}{cz+d} = A + \frac{B}{z+C}$$

We reach the expression $A + \frac{q}{cz+d}$ through the process of long division. The last expression follows by division of the numerator and denominator by c. We have replaced the results of expressions without z using A, B, C. Once we have the function in this form, notice that

$f = f_3 \circ f_2 \circ f_1$, where

$$f_1 : z \mapsto z + C =: z_1$$
$$f_2 : z_1 \mapsto \frac{B}{z_1} =: z_2$$
$$f_3 : z_2 \mapsto z_2 + A$$

Observing that f_1 and f_3 are just translations (which will not change the shape of the domain), it suffices to show that $z \mapsto \frac{1}{z}$ maps generalized circles to generalized circles.
Let $z = x + iy$. Recall that we can write the real and imaginary parts as $x = \frac{z+\overline{z}}{2}, y = \frac{z-\overline{z}}{2i}$.
We describe the equation of a line and of a circle, and then we describe our transformation.
The equation of a line is $ax + by + c = 0$. We can substitute x and y to get

$$a \cdot \frac{z+\overline{z}}{2} + b \cdot \frac{z-\overline{z}}{2i} + c = 0$$
$$\underbrace{\frac{1}{2}(a-bi)}_{A} z + \underbrace{\frac{1}{2}(a+bi)}_{\overline{A}} \overline{z} + c = 0$$
$$Az + \overline{A}\overline{z} + c = 0$$

for some $A \in \mathbb{C}$ and $c \in \mathbb{R}$. This is still the equation of a line. Turning to the equation of a circle

$$(x-a)^2 + (y-b)^2 - r^2 = 0$$
$$x^2 + y^2 - 2ax - 2by - r^2 = 0$$
$$z\overline{z} + Az + \overline{A}\overline{z} + c = 0$$

for some $A \in \mathbb{C}$ and $c \in \mathbb{R}$. We have asserted that $-2ax - 2by - r^2 = Az + \bar{A}\bar{z} + c$ because we have established, immediately prior, that any homogeneous linear expression in x and y with real coefficients may be represented in this way.
Now, consider $z \mapsto \frac{1}{z} = w$. So $z = \frac{1}{w}$ and we want to substitute this into the equation of a circle that we found above, in order to determine the image.

$$\frac{1}{w\overline{w}} + \frac{A}{w} + \frac{\overline{A}}{\overline{w}} + c = 0$$
$$cw\overline{w} + \overline{A}w + A\overline{w} + 1 = 0$$

In the case where $c = 0$, we have a line.
In the case where $c \neq 0$, we may divide by c to obtain the equation

$$w\bar{w} + \frac{\bar{A}}{c}w + \frac{A}{c}\bar{w} + \frac{1}{c} = 0$$

Since $c \in \mathbb{R}$, this means that the second and third terms are conjugates, hence, we have the equation of a circle.

□

Example. Let $f(z) = \frac{z-i}{z+i}$.
Let $L = \{z \in \mathbb{C} : \operatorname{Im} z = 0\}$. What is $f(L)$?
Let $H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and let f be the same, what is $f(H)$?

To approach this problem, we want to consider some representative points. We see that $f(0) = \frac{-i}{i} = -1, f(\infty) = 1, f(1) = \frac{1-i}{1+i} = -i$. So as we move along the real line, it appears that the image traces out the path of the unit circle. We also see that $|f(z)| = 1$, so that $f(z)$ belongs to the unit circle. Hence

$$f(L) = \{w \in \mathbb{C} : |w| = 1\}$$

For the second question, notice that

$$z \in H \iff |z-i| < |z+i| \iff \left|\frac{z-i}{z+i}\right| < 1$$

Geometrically, if z is in the upper half plane, it is closer to i than to $-i$. Thus,

$$f(H) = \{w \in \mathbb{C} : |w| < 1\}$$

5 October 11, 2024

We begin with some details on simple ways to understand certain linear fractional mappings. Suppose we have the set $C = \{z \in \mathbb{C} : |z-1| = 1\}$ and $f(z) = \frac{1}{z-2}$. Since we know $2 \in C$, it

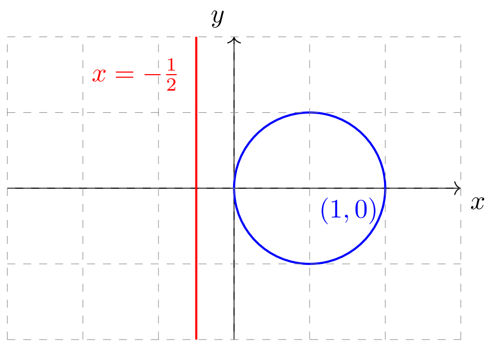


Figure 4: C and $f(C)$

follows that $\infty \in f(C)$. If ∞ lies in the image of a linear fractional mapping, we know that the image must be a line. Also, we know that linear fractional mappings preserve angles. Since C is orthogonal to the real axis at $z = 0$, the preservation of angles implies that $f(C)$ is orthogonal to the image of the real axis under f (this image is simply the real axis again), at $f(0)$. Plugging in gives us $f(0) = -\frac{1}{2}$. This is enough information to determine that

$$f(C) = \left\{ w \in \mathbb{C} : \operatorname{Re} w = -\frac{1}{2} \right\}$$

5.1 More on Linear Fractional Transformations

Proposition. Suppose that (a_1, b_1, c_1) and (a_2, b_2, c_2) are two triples of distinct points of $\overline{\mathbb{C}}$. Then there exists a unique linear fractional transformation f such that

$$f(a_1) = a_2, \quad f(b_1) = b_2, \quad f(c_1) = c_2$$

Proof. First, we find a linear fractional mapping such that $f(a_1) = 0, f(b_1) = 1, f(c_1) = \infty$. Observe that

$$z \mapsto \frac{z - a_1}{z - c_1}$$

satisfies the first and third condition. Now, we may choose some nonzero λ such that $f(b_1) = 1$. Hence, the desired linear fractional mapping is

$$z \mapsto \lambda \cdot \frac{z - a_1}{z - c_1}$$

Remark. A composition of two linear fractional transformations is again a linear fractional transformation. The inverse to a linear fractional transformation is again a linear fractional

transformation.
Consider the following diagram.

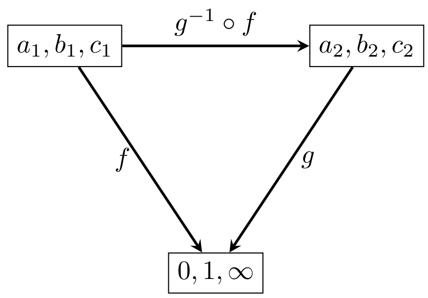


Figure 5: Existence of the claimed transformation

Based on our remark, the mapping $g^{-1} \circ f$ is linear fractional and existence follows.
Now, to prove uniqueness, we begin with a particular case. Let $f(z) = \frac{pz+q}{rz+s}$ and $f(0) = 0$, $f(1) = 1$, and $f(\infty) = \infty$.

- Since $f(\infty) = \infty$, this means that finite numbers cannot map to ∞. Hence $r = 0$ and without loss of generality, we may take $s = 1$. With this, our function becomes $f(z) = pz + q$.
- Continuing, $f(0) = 0$ shows us that $q = 0$ and $f(1) = 1$ shows us that $p = 1$.

Substituting all of these values in, we have $f(z) = z$. In other words, with these conditions, f must be the identity function.
To proceed with the general case for uniqueness, suppose f_1 and f_2 are linear fractional mappings that satisfy the three conditions in the statement of the proposition. Consider the following diagram. By what we have proven earlier about the uniqueness in a particular

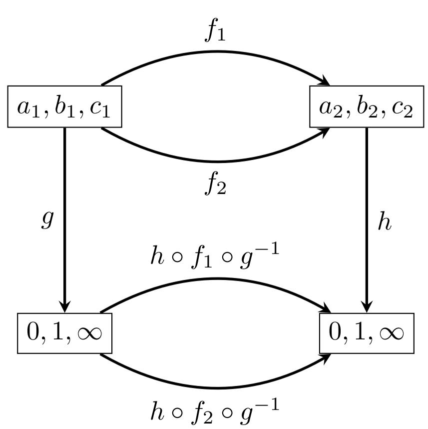


Figure 6: Uniqueness of the transformation

case, we see that $h \circ f_1 \circ g^{-1} = h \circ f_2 \circ g^{-1}$. Hence $f_1 = f_2$, so the mapping is unique.

$\square$

5.2 Linear Fractional Automorphisms

We will consider linear fractional mappings on and onto the upper half plane $H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.

Proposition. A linear fractional mapping f is a bijection $H \to H$ if and only if $f(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ and $\begin{vmatrix} a & b \\ c & d \end{vmatrix} > 0$. Note that this is equivalent to the condition that $f(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ and $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1$, if we scale the first f by its determinant.

Proof. Consider the following figure, where we understand the upper half plane as a hemisphere of the Riemann sphere.

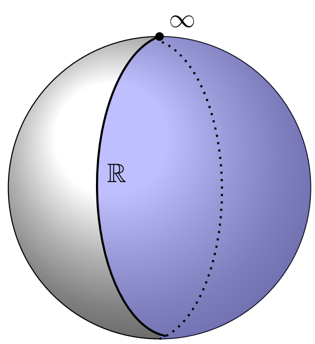


Figure 7: Representation of H on the Riemann sphere

Above, the real line $\mathbb{R} \cup \{\infty\}$ is represented as a meridian (or a circle) while the region in blue is the upper half plane, H. If f is a linear fractional map, then $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a bijection. Suppose we have a bijection $f : H \to H$ as well. Because the inverse of a linear fractional mapping is a linear fractional mapping, f is a homeomorphism (bijection with continuous inverse).

Aside. Let X, Y be topological spaces, $f : X \to Y$ a homeomorphism, and $A \subseteq X$ a subspace. Then

1. $f(A^\circ) = f(A)^\circ$
2. $f(\partial A) = \partial f(A)$

As a result, f maps the boundary of H to the boundary of H, that is

$$f(\mathbb{R} \cup \{\infty\}) = \mathbb{R} \cup \{\infty\}$$

Since $f(z) = \frac{az+b}{cz+d}$, this property of mapping the boundary to the boundary means we must have $a, b, c, d \in \mathbb{R}$. In addition, observe that

$$f(H) = H \iff f(i) \in H \iff \operatorname{Im} f(i) > 0$$

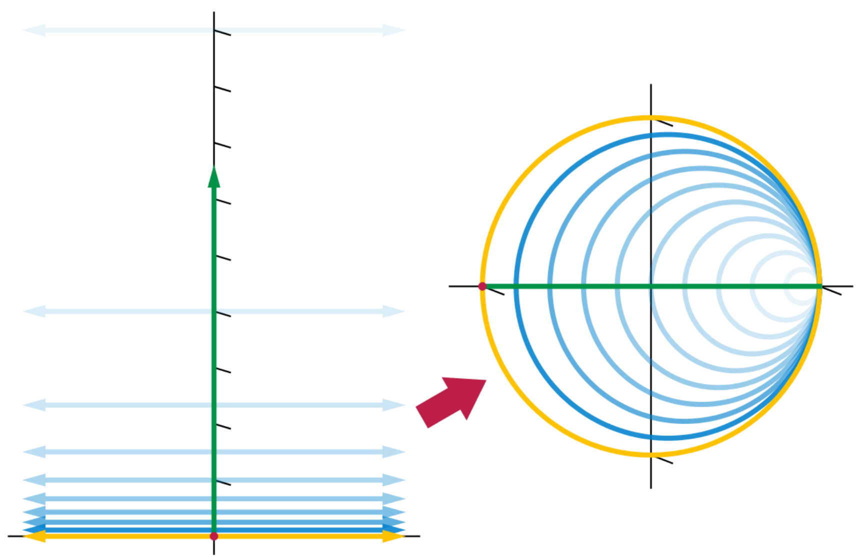
We may compute

$$\begin{aligned} 0 < \operatorname{Im} f(i) &= \operatorname{Im} \frac{ai+b}{ci+d} = \operatorname{Im} \frac{(ai+b)(-ci+d)}{c^2+d^2} \\ &= \frac{ad-bc}{c^2+d^2} \\ &= \frac{1}{c^2+d^2} \cdot \begin{vmatrix} a & b \\ c & d \end{vmatrix} \end{aligned}$$

□

5.3 Linear Fractional Automorphisms of the Disk

Let $D = \{z \in \mathbb{C} : |z| < 1\}$, the unit disk, and define $F(z) = \frac{z-i}{z+i}$. This mapping is called the **Cayley transform**. This mapping allows us to go from the upper half plane to the unit disk.



Consider the commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{F} & D \\ \downarrow{g} & & \downarrow{f} \\ H & \xrightarrow{F} & D \end{array}$$

We may write $f = F \circ g \circ F^{-1}$, where $g(z) = \frac{az+b}{cz+d}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$.

We can give a more direct characterization of these automorphisms as well. Suppose the following conditions hold:

1. Let $\varphi \in \mathbb{R}$, then $z \mapsto e^{i\varphi} \cdot z$ is an automorphism of D, as it is a rotation.
2. $|a| < 1$.

Then

$$f(z) = \frac{z-a}{1-\overline{a}z}$$

describes a linear fractional automorphism of the disk.

Proposition. Given f as above, we have $f(D) = D$.

Proof. Suppose $1 - \overline{a}z = 0$. Solving for z gives $z = \frac{1}{\overline{a}}$. Also, $\left|\frac{1}{\overline{a}}\right| = \frac{1}{|a|} > 1$. Since $|z| < 1$, being that we are on the unit disk, this means we must have $f(D) \subseteq \mathbb{C}$. Now, if we prove that $f(\partial D) = \partial D$, then we are done. [Why?]

To prove this, it is sufficient to show that $|z| = 1 \implies |f(z)| = 1$. Note that $z\overline{z} = |z| = 1$, so that $\frac{1}{z} = \overline{z}$. Observe that

$$\begin{aligned} |f(z)| &= \left|\frac{z-a}{1-\overline{a}z}\right| = |z| \cdot \left|\frac{1-az^{-1}}{1-\overline{a}z}\right| \\ &= \frac{|1-a\overline{z}|}{|1-\overline{a}z|} \end{aligned}$$

But notice that $\overline{1-a\overline{z}} = 1 - \overline{a}z$, so these numbers have the same moduli, showing that $|f(z)| = 1$.

□

Above, we have shown that if $\varphi \in \mathbb{R}$ and $|a| < 1$, then

$$z \mapsto e^{i\varphi} \cdot \frac{z-a}{1-\overline{a}z}$$

is an automorphism of D. Actually, it turns out that any linear fractional automorphism of D is of this form.

Remark. Any holomorphic bijection $f : H \to H$ is linear fractional. The same applies to holomorphic bijections $D \to D$. We will prove this later.

5.4 Conformal Mappings

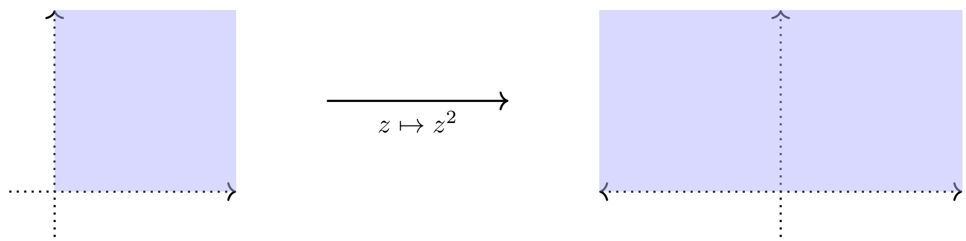
Definition. *Conformal Mapping*

Suppose that $U, V \subseteq \mathbb{C}$ are open subsets. A conformal mapping $f : U \to V$ is a mapping that is holomorphic and bijective.

Remark. It will follow from general theory that if f is a conformal mapping, then $f'(z) \neq 0$ for any $z \in U$ and $f^{-1} : V \to U$ is also holomorphic. Where is this from?

Example. Let $U = \{z \in \mathbb{C} : \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$. Construct a conformal mapping $f : U \to H = \{w \in \mathbb{C} : \operatorname{Im} w > 0\}$.

The map $z \mapsto z^2$ works for this. We proved that $f(z) = z^2$ is holomorphic using the limit



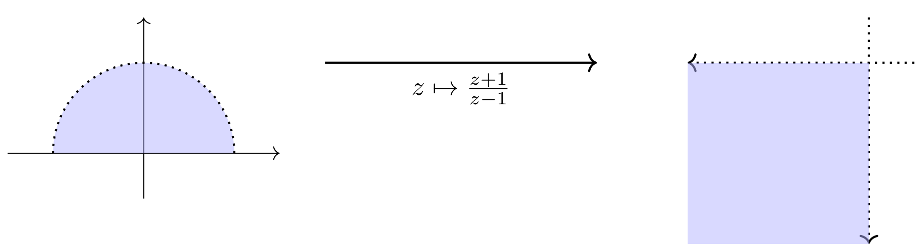
definition in a prior class. To see that it is surjective, take some $h \in H$. We may write $h = re^{i\varphi}$. But then $\sqrt{r}e^{i\frac{\varphi}{2}}$ is mapped to h by f. To see that it is injective, suppose that $f(z_1) = f(z_2)$. This implies that $(z_1 + z_2)(z_1 - z_2) = (z_1^2 - z_2^2) = 0$. We must have $z_1 = z_2$, since $z_1 = -z_2$ means either $z_1 \notin U$ or $z_2 \notin U$.

Example. Let $U = \{z \in \mathbb{C} : |z| < 1, \operatorname{Im} z > 0\}$. Determine a conformal mapping from U to H.

We want to think about the boundary and the form of a linear fractional mapping. First, we want to reduce this to the first problem, we can do this by mapping the upper half disk to the open third quadrant by

$$f_1(z) = \frac{z+1}{z-1}$$

We know that this function works because we chose values so that $f(1) = \infty$, $f(-1) = 0$. We also see that $f(0) = -1$ and $f(i) = \frac{i+1}{i-1} = -i$. This is enough to show that the segment from $(-1, 1)$ is mapped to $(0, 1)$ and the upper unit semicircle is mapped to the negative imaginary axis. From here, we can just take $f_2(z) = z^2$ to obtain H. Hence, our mapping



will be $f(z) = (f_2 \circ f_1)(z) = \left(\frac{z+1}{z-1}\right)^2$

Example. Let $U = H \setminus (0, i]$. Find a conformal mapping from U to H.

The difficulty here is the irregular shape of the boundary in U. Luckily, setting $f_1(z) = z^2 =: z_1$, we notice that $f_1(U) = \mathbb{C} \setminus [-1, \infty)$. As a result, we are able to translate the image into something more familiar. Taking $f_2(z_1) = z_1 + 1 =: z_2$ and $f_3(z_2) = \sqrt{z_2}$, with $\sqrt{-1} = i$ as the branch, our conformal mapping will be

$$f(z) = (f_3 \circ f_2 \circ f_1)(z) = \sqrt{z^2 + 1}, \quad \sqrt{-1} = i$$

Note that a translation is a conformal mapping.

Example. Let $U = \{z \in \mathbb{C} : |z| > 1\}$ and let $f : U \to \mathbb{C}$ be given by

$$z \mapsto z + \frac{1}{z}$$

1. Describe the image $f(U)$.
2. Is $f : U \to U$ one-to-one?

Recall that

$$f(U) = \left\{ a \in \mathbb{C} \mid \exists z \in U : z + \frac{1}{z} = a \right\}$$

Since $z \in U$, we must have $|z| > 1$. Now, we may write

$$\frac{z^2 + 1}{z} = a \implies z^2 - za + 1 = 0$$

Using Vieta's formulas, we know that for roots z_1, z_2 to this equation, we have $z_1 + z_2 = \frac{a}{1}$ and $z_1 z_2 = 1$. Since $|z| > 1$, the second equation shows us that $|z_1| > 1$ or $|z_2| > 1$. A comment on what is happening here: we are determining the solutions to $z + \frac{1}{z} = a$ and then excluding the ones that are not in U. As a more concrete condition, we consider the complement $\mathbb{C} \setminus f(U)$. Notice that

$$\begin{aligned} \mathbb{C} \setminus f(U) &= \left\{ a \in \mathbb{C} \mid \forall z \in U : z + \frac{1}{z} \neq a \right\} \\ &= \{a \in \mathbb{C} : |z_1| = 1 \text{ and } |z_2| = 1\} \end{aligned}$$

In the complement $\mathbb{C} \setminus f(U)$, we want roots z_1, z_2 so that $z_1, z_2 \notin U$. This is only possible when $|z_1| = 1$ and $|z_2| = 1$. Since $|z_1| = 1$, we may write $z_1 = \cos(t) + i\sin(t) = e^{it}$. Substituting, we have

$$a = e^{it} + e^{-it} = 2\cos t,\ t \in \mathbb{R}$$

Hence $\mathbb{C} \setminus f(U) = [-2, 2]$. Finally, we see that $f(U) = \mathbb{C} \setminus [-2, 2]$.

The second question was unanswered the class period. It is asking if $|z| > 1$ and $z + \frac{1}{z} = a$ implies a unique solution.

6 October 18, 2024

6.1 Symmetry with respect to a Generalized Circle

Suppose $C \subseteq \overline{\mathbb{C}}$ is a generalized circle and $a, b \in \overline{\mathbb{C}}$.

Definition. *Symmetry wrt. a Generalized Circle*
One says that a and b are symmetric with respect to C if any generalized circle passing through a and b is orthogonal to C. By convention, if $a \in C$, then a is symmetric to itself. Why does this convention make sense?

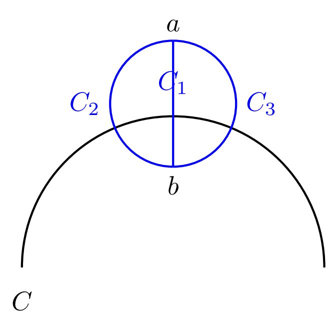


Figure 8: Symmetric points a and b to the circle C

We chose the points $\frac{3}{2}i$ and $\frac{2}{3}i$ for the image above. The circle is centered at the origin with radius 1.

Example. Let C be a line. Then a and b are symmetric with respect to C if and only if a and b are symmetric with respect to C in the usual sense.

So, if C is a line, the existence and uniqueness of the symmetric point is obvious. Letting C be the imaginary axis and given a point z, notice that $-\operatorname{Re} z + i \operatorname{Im} z = -\overline{z}$ will be such a symmetric point. Further, z and its conjugate $\overline{z}$ are symmetric with respect to the real axis.

Example. Let $C = \{z \in \mathbb{C} : |z| = 1\}$. What point $b \in \overline{\mathbb{C}}$ is symmetric to 0 with respect to C?

It turns out that ∞ is the point that will be symmetric to 0.

Proposition. For any generalized circle C and any $a \in \overline{\mathbb{C}} \setminus C$, there exists a unique b that is symmetric to a.

Proof. Let f be a linear fractional transformation such that $f(C) = L$, where L is a line. Such a transformation must exist once we choose three points which determine C and map them to three points on L. We claim that a is symmetric to b with respect to c if and only if $f(a)$ is symmetric to $f(b)$ with respect to $f(c)$.
Suppose a and b are symmetric with respect to C. Since f is a linear fractional transformation, it is a conformal mapping. Linear fractional transformations also map generalized circles to generalized circles. Since f is conformal, it has a holomorphic inverse f^{-1} and this inverse will also have the form of a linear fractional transformation. Suppose C_1 is a generalized circle passing through $f(a)$ and $f(b)$. Then $f^{-1}(C_1)$ will be a generalized circle passing through a and b. Since a and b are symmetric with respect to C, $f^{-1}(C_1)$ is orthogonal to C. Since f is conformal and preserves angles, $f(f^{-1}(C_1)) = C_1$ is orthogonal to $f(C)$. Now, the reverse direction may be solved in the same way by considering f^{-1} in place of f and the appropriate images of points and generalized circles.

□

I need to double check proof above because I filled in the details

Example. Let $C = \{z \in \mathbb{C} : z < 1\}$ and $|a| < 1$. Find the point that is symmetric to a with respect to C.

To begin, we want to consider mapping C to some line. We have a choice, so we will pick the real axis to simplify things. By picking the real axis, we know that the symmetric point in the image will be the conjugate.
We can determine a linear fractional transformation by choosing three points on the unit circle and mapping them to the real line. Let us choose $f(-1) = 0$ and $f(1) = \infty$, our transformation will then have the form $f(z) = b \cdot \frac{z+1}{z-1}$ Next, we compute $f(i) = \frac{i+1}{i-1} = -i$. We need $f(i)$ to map to a real number, so for convenience, we can choose $f(i) = 1$. Then, we see that $b = -\frac{1}{i} = i$. So, our desired transformation is

$$f(z) = i \cdot \frac{z+1}{z-1}$$

So, $f(a) = i\left(\frac{a+1}{a-1}\right)$ and further, we know

$$S_L(f(a)) = \overline{f(a)} = \overline{i \cdot \left(\frac{a+1}{a-1}\right)} = -i \cdot \frac{\overline{a}+1}{\overline{a}-1}$$

Now, we need to find the inverse function in order to determine the symmetric point in the domain. We may identify linear fractional mappings with matrices, so

$$f(z) = \frac{iz+i}{z-1} \sim \begin{pmatrix} i & i \\ 1 & -1 \end{pmatrix} \implies f^{-1} \sim \begin{pmatrix} -1 & -i \\ -1 & i \end{pmatrix} \sim \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

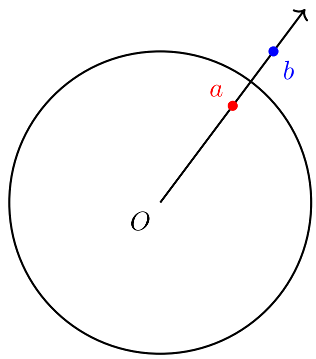


Figure 9: The symmetric points a, $\frac{1}{\overline{a}}$

Note that we can drop and multiply by constants because multiplying the numerator and denominator by the same constant does not change the linear fractional mapping. This is why we can drop the determinant and multiply by -1.
So the inverse function is $f^{-1}(w) = \frac{w+i}{w-i}$. We compute

$$f^{-1}(\overline{f(a)}) = \frac{-i \cdot \frac{\overline{a}+1}{\overline{a}-1} + i}{-i \cdot \frac{\overline{a}+1}{\overline{a}-1} - i} = \frac{-(\overline{a}+1) + (\overline{a}-1)}{-(\overline{a}+1) - (\overline{a}-1)} = \frac{1}{\overline{a}}$$

This shows us that $|a| \cdot |b| = |a| \cdot \frac{1}{|\overline{a}|} = 1$. Also, $\arg \overline{a} = -\arg a$ and $\arg\left(\frac{1}{\overline{a}}\right) = \arg a$.

6.2 Full Description of Linear Fractional Automorphisms of the Disk

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and $C = \{z \in \mathbb{C} : |z| = 1\}$. Suppose $f(D) = D$ is a linear fractional transformation. Since f maps points that are symmetric with respect to C to points that are symmetric with respect to C, we see that $f(C) = C$. Recall that points on C are symmetric with respect to themselves. Now, there exists a point $a \in D$ satisfying $f(a) = 0$. From our previous computation, we know that it is symmetric to $\frac{1}{\overline{a}}$. We also discussed that 0 is symmetric to ∞. It follows that $f(\frac{1}{\overline{a}}) = \infty$.
Given the above information and the form

$$f(z) = \frac{pz+q}{rz+s}$$

we know that the numerator vanishes for $z = a$ and the denominator for $z = \frac{1}{\overline{a}}$. So,

$$f(z) = c \cdot \frac{z-a}{1-\overline{a}z}$$

By a previous proposition, we know that $f(z) = \frac{z-a}{1-\overline{a}z}$ maps D to D. So, $|c| = 1$ and $c = e^{i\theta}$ for $\theta \in \mathbb{R}$. Finally, we have

$$f(z) = e^{i\theta} \cdot \frac{z-a}{1-\overline{a}z}$$

6.3 Cauchy's Theorem

Theorem. *Cauchy's Integral Theorem (Cauchy-Goursat)*
Informally, let $f : \overline{D} \to \mathbb{C}$ be continuous and let f be holomorphic on D. Then,

$$\int_\gamma f(z)\,dz = 0$$

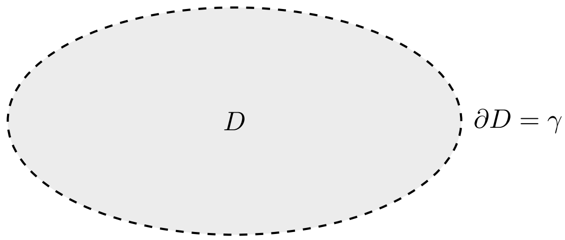


Figure 10: Region of Integration and Boundary

Recall Green's formula

$$\int_\gamma P(x,y)\,dx + Q(x,y)\,dy = \iint_U \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Now, letting $z = x + iy$ and $dz = dx + i\,dy$, we have

$$f(z)\,dz = (u + iv)(dx + i\,dy) = u\,dx - v\,dy + i(u\,dy + v\,dx)$$
$$\int_\gamma f(z)\,dz = \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (v\,dx + u\,dy)$$

Invoking Green's theorem, we see that

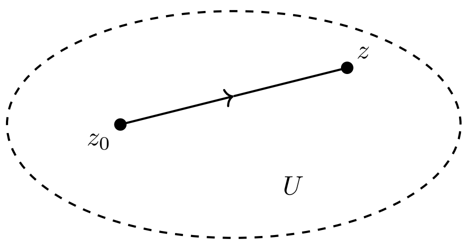
$$\int_\gamma (u\,dx - v\,dy) = \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$
$$\int_\gamma (v\,dx + u\,dy) = \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

The right-hand side of both expressions above are identically zero due to Cauchy-Riemann.

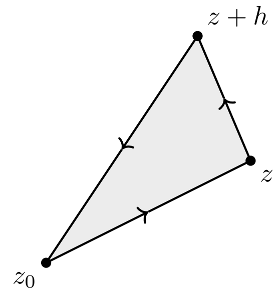
Aside. Proofs of Cauchy's formula may be found in many textbooks. T. Gamelin's *Complex Analysis* and W. Rudin's *Real and Complex Analysis* are examples. Also, to invoke Green's theorem, we need to assume the continuity of the partial derivatives. The formal proof is much more difficult because of the absence of this assumption, however, the continuity of the partials actually follows as a consequence of Cauchy's integral formula.

Proposition. *Existence of a primitive.* Suppose $U \subseteq \mathbb{C}$ is open and convex, and that $f : U \to \mathbb{C}$ is holomorphic. Then there exists an $F : U \to \mathbb{C}$ such that F is holomorphic on U and $F'(z) = f(z)$ for any $z \in U$.

Proof. Fix a point $z_0 \in U$.



(a) A path in U



(b) Boundary and Area of Δ

For any $z \in U$, put

$$F(z) := \int_{[z_0,z]} f(z)\, dz$$

How do we compute F'? First, we want to determine $F(z+h) - F(z)$. Since U is convex, this means that the triangle is contained in U.
By Cauchy's theorem, we know

$$\int_{\partial\Delta} f(\zeta)\, d\zeta = 0$$

We compute

$$\underbrace{\int_{[z_0,z]} f(\zeta)\, d\zeta}_{F(z)} + \int_{[z,z+h]} f(\zeta)\, d\zeta \underbrace{- \int_{[z_0,z+h]} f(\zeta)\, d\zeta}_{-F(z+h)} = 0$$

$$\begin{aligned} F(z+h) - F(z) &= \int_{[z,z+h]} f(\zeta)\, d\zeta \\ &= \underbrace{\int_{[z,z+h]} f(z)\, d\zeta}_{h \cdot f(z)} + \underbrace{\int_{[z,z+h]} (f(\zeta) - f(z))\, d\zeta}_{I} \end{aligned}$$

Observe that via the trivial estimate

$$|I| \leq \sup_{I \in [z,z+h]} |f(\zeta) - f(z)| \cdot |h| \xrightarrow[h \to 0]{} 0$$

Since f is holomorphic, it is continuous, and since f is continuous, it is bounded. This justifies the above limit. So,

$$\frac{F(z+h) - F(z)}{h} = f(z) + \frac{I}{h}$$

Taking the limit of both sides as $h \to 0$, we see that $F'(z) = f(z)$.

□

Corollary. If $U \subseteq \mathbb{C}$ is open and convex, $f : U \to \mathbb{C}$ is holomorphic, then $\int_\gamma f(t)\,dt = 0$ whenever γ is a closed path in U.

6.4 Cauchy's Integral Formula

Theorem. *The Cauchy Formula*
Suppose D is an open subset in $\mathbb{C}$ bounded by a piecewise smooth curve and $f : \overline{D} \to \mathbb{C}$ is continuous and holomorphic on D. Then, for any $a \in D$, one has

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta - a}$$

Remark. We informally understand the orientation of γ to be in the direction such that D is on our left as we travel along γ.
Rigorously, regard $\overline{D}$ as a submanifold of $\mathbb{C}$ with boundary. Then $\mathbb{C}$ has a canonical orientation (the basis $(1, i)$ is positive) and D and $\partial D = \gamma$ have the induced orientation.
Proof. We prove Cauchy's Formula. Let

$$D_\varepsilon = D \setminus \{z \in \mathbb{C} : |z - a| < \varepsilon\} \quad \text{and} \quad \gamma_\varepsilon = \{z \in \mathbb{C} : |z - a| = \varepsilon\}$$

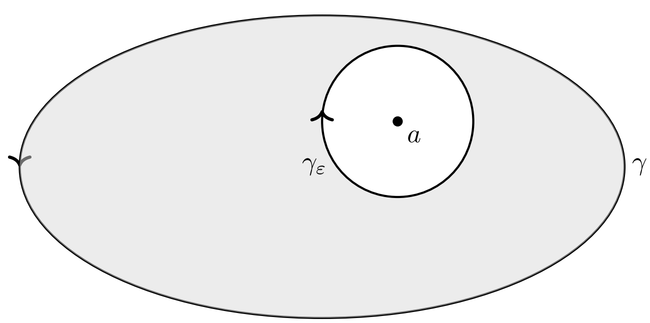


Figure 12: The Region D_ε

We apply Cauchy's theorem to D_ε and $\frac{f(z)}{z-a}$ and see

$$0 = \int_{\partial D_\varepsilon} \frac{f(z)\,dz}{z - a} = \int_\gamma \frac{f(z)\,dz}{z - a} - \int_{\gamma_\varepsilon} \frac{f(z)\,dz}{z - a}$$

So that $\int_\gamma \frac{f(z)\,dz}{z-a} = \int_{\gamma_\varepsilon} \frac{f(z)\,dz}{z-a}$. Since f is holomorphic, we may write

$$f(z) = f(a) + f'(a)(z - a) + \varphi(z) = f(a) + R(z)$$

where $\lim_{z\to a}\left|\frac{\varphi(z)}{z-a}\right| = 0$. Define $R(z) = f'(a)(z-a) + \varphi(z)$. Since $\varphi(z) = o(z-a)$ and $f'(a)(z-a) \le C|z-a|$ for some constant, we see that $|R(z)| \le C|z-a|$ for z sufficiently close to a. Now, we have

$$\int_{\gamma_\varepsilon} \frac{f(z)\,dz}{z-a} = \underbrace{\int_{\gamma_\varepsilon} \frac{f(a)\,dz}{z-a}}_{2\pi i\cdot f(a)} + \underbrace{\int_{\gamma_\varepsilon} \frac{R(z)\,dz}{z-a}}_{I(\varepsilon)}$$

Observe that by using the trivial estimate,

$$|I(\varepsilon)| \le 2\pi\varepsilon \cdot \sup_{z\in\gamma_\varepsilon} \frac{|R(z)|}{|z-a|} \le 2\pi\varepsilon \cdot C$$

By shrinking the inner circle, we have $|I(\varepsilon)| \xrightarrow[\varepsilon\to 0]{} 0$. Finally

$$\int_{\gamma} \frac{f(\zeta)\,d\zeta}{\zeta-a} = \int_{\gamma_\varepsilon} \frac{f(\zeta)\,d\zeta}{\zeta-a} = 2\pi i \cdot f(a)$$

□

It is important to notice that Cauchy's integral formula allows us to determine points on the interior using a calculation involving points on the boundary.

6.5 Cauchy-Type Integrals

Proposition. *Regularity condition.* Suppose γ is a piecewise smooth path in $\mathbb{C}$, $f : |\gamma| \to \mathbb{C}$ is continuous, and $n > 0$ is an integer. Put

$$\varphi(z) = \int_{\gamma} \frac{f(\zeta)\,d\zeta}{(\zeta-z)^n}$$

This function is well-defined for $z \notin |\gamma|$, so $\varphi : \mathbb{C}\setminus|\gamma| \to \mathbb{C}$. Then φ is holomorphic on $\mathbb{C}\setminus|\gamma|$ and

$$\varphi'(z) = n \cdot \int_{\gamma} \frac{f(\zeta)\,d\zeta}{(\zeta-z)^{n+1}}$$

Informally, we may say

$$\frac{d}{dz}\left(\frac{f(z)}{(\zeta-z)^n}\right) = \frac{n\cdot f(z)}{(\zeta-z)^{n+1}}$$

We may think of the above proposition as a case where we can interchange the integral and the derivative.

Proof. Considering the difference $\varphi(z+h) - \varphi(z)$, we see that the integrand is

$$f(\zeta)\left(\frac{1}{(\zeta-(z+h))^n} - \frac{1}{(\zeta-z)^n}\right)$$

Computing the difference of the fractions, we have

$$\frac{1}{(\zeta-(z+h))^n}-\frac{1}{(\zeta-z)^n}=\frac{(\zeta-z)^n-(\zeta-z-h)^n}{(\zeta-z-h)^n(\zeta-z)^n}$$
$$=\frac{(\zeta-z)^n-(\zeta-z)^n+n(\zeta-z)^{n-1}\cdot h+h^2\cdot P(z,\zeta,h)}{(\zeta-z-h)^n(\zeta-z)^n}$$

where we have used the binomial theorem to expand $((\zeta-z)-h)^n$ and where $P(z,\zeta,h)$ is a 'universal polynomial'. Making substitutions, we have

$$\frac{\varphi(z+h)-\varphi(z)}{h}=\int_\gamma\frac{nf(\zeta)\,d\zeta}{(\zeta-z-h)^n(\zeta-z)}+h\int_\gamma\frac{P(z,\zeta,h)\,dz}{(\zeta-z-h)^n(\zeta-z)^n}$$

As $h\to 0$, this tends to $(\zeta-z)^{n+1}$ (bounded from 0) uniformly for $\zeta\in|\gamma|$. So, in the first integral, the integral converges uniformly and the first term converges to

$$\int_\gamma\frac{nf(\zeta)\,d\zeta}{(\zeta-z)^{n+1}}$$

In the second term, the integrand converges uniformly, this implies the integral converges to something finite. Hence, the second term tends to 0. Thus

$$\varphi'(z)=\lim_{h\to 0}\frac{\varphi(z+h)-\varphi(z)}{h}=n\cdot\int_\gamma\frac{f(\zeta)\,d\zeta}{(\zeta-z)^{n+1}}$$

□

Observe that the above proposition shows us that if a function satisfies Cauchy's integral formula, then it is holomorphic. The key point is that if a function satisfies Cauchy's integral formula, we are able to compute its derivative explicitly, so it is holomorphic. See Theorem 1.1 in The Fundamental Theorem in Complex Function Theory for a more rigorous elaboration.

7 November 1, 2024

7.1 Applications of Cauchy-Type Integrals

Let $D \subseteq \mathbb{C}$ be a bounded domain with a piecewise smooth boundary and let $f : \overline{D} \to \mathbb{C}$ be continuous and holomorphic on D. Cauchy's integral formula says

$$z \in D \implies f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

So, taking the derivative, we have

$$f'(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)\,d\zeta}{(\zeta - z)^2}$$

Since f' is defined by a Cauchy-Type integral, applying the proposition once more shows that f' is holomorphic.

Corollary. If $U \subseteq \mathbb{C}$ is open and $f : U \to \mathbb{C}$ is holomorphic, then $f' : U \to \mathbb{C}$ is also holomorphic (hence, so are the higher-order derivatives).

Note that U should have a piecewise smooth boundary. However, since U is open and since holomorphicity is a local property, we can just restrict f to a disk around the specific point so that we can apply our proposition.

If D is bounded with a piecewise smooth boundary, then

$$f''(z) = \frac{2}{2\pi i}\int_{\partial D} \frac{f(\zeta)\,dz}{(\zeta - z)^3}, \quad f'''(z) = \frac{3!}{2\pi i}\int_{\partial D} \frac{f(\zeta)\,dz}{(\zeta - z)^4}$$
$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\partial D} \frac{f(\zeta)\,dz}{(\zeta - z)^{n+1}}$$

Theorem. *Morera's Theorem*
Suppose that $U \subseteq \mathbb{C}$ is open and convex, $f : U \to \mathbb{C}$ is continuous, and for any solid triangle $\Delta \subseteq U$

$$\int_{\partial\Delta} f(z)\,dz = 0$$

Then f is holomorphic. This is a partial converse to Cauchy's theorem.

Proof. We need to show f' exists on U. Fix $z_0 \in U$. Put $F(z) = \int_{[z_0,z]} f(\zeta)\,d\zeta$. Then F is holomorphic and $F' = f$ for any $z \in U$, by a previous proposition. By the regularity of holomorphic functions (we know F is infinitely differentiable), it follows that f is holomorphic. □

Alternative. Another way to proceed is to justify again the fact that $F' = f$.

$$\int_{\partial\Delta} f(\zeta)\,d\zeta = 0 \implies F(a+h) - F(a) = \int_{[a,a+h]} f(\zeta)\,d\zeta$$

Dividing through by h, we have

$$\begin{aligned}\frac{F(a+h)-F(a)}{h} &= \frac{\int_{[a,a+h]} f(\zeta)\,d\zeta}{h} \\ &= \frac{\int_{[a,a+h]}(f(a)+o(1))\,d\zeta}{h} \\ &= f(a)+\frac{1}{h}\int_{[a,a+h]} o(1)\,d\zeta\end{aligned}$$

The continuity of f justifies this use of asymptotic notation. Given any $\varepsilon > 0$, the integrand satisfies $|o(1)| < \varepsilon$ for sufficiently small h, so at most the value of the integrand is $\varepsilon|h|$. This means the result will be arbitrarily (within ε) close to $f(a)$, so $F' = f$ and we use the regularity property to finish the proof.

7.2 Expanding Holomorphic Functions into Series

We define

$$\begin{aligned}\overline{D}_{a,r} &= \{z \in \mathbb{C} : |z-a| \leq r\} \\ D_{a,r} &= \operatorname{Int}(\overline{D}_{a,r}) = \{z \in \mathbb{C} : |z-a| < r\}\end{aligned}$$

Theorem.
If f is holomorphic on $D_{a,r}$, then, in $D_{a,r}$, one has the expansion

$$\begin{aligned}f(z) &= c_0 + c_1(z-a) + c_2(z-a)^2 + \ldots + c_n(z-a)^n + \ldots \\ &= \sum_{k=0}^{\infty} c_k z^k\end{aligned}$$

The series in the right-hand side converges uniformly on any compact set $K \subseteq D_{a,r}$.

Proof. It suffices to check that this series converges to $f(z)$ at any $z \in D_{a,r}$. Pick a p

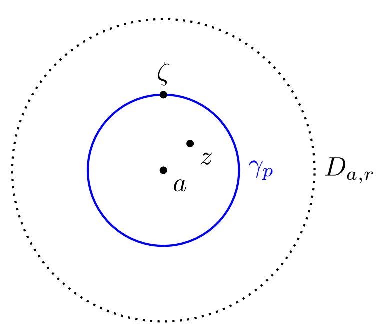


Figure 13: Visual setup of objects in the proof

satisfying $|z-a| < p < r$ and define $\gamma_p = \{\zeta \in \mathbb{C} : |\zeta - a| = p\}$. Since f is holomorphic, by

Cauchy's integral formula we have

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_p}\frac{f(\zeta)\,d\zeta}{\zeta - z}$$

We expand $\frac{1}{\zeta - z}$, for $\zeta \in \gamma_p$:

$$\begin{aligned}\frac{1}{\zeta - z} &= \frac{1}{(\zeta - a) - (z - a)} = \frac{1}{\zeta - a}\cdot\frac{1}{1 - \frac{z-a}{\zeta-a}} \\ &= \frac{1}{\zeta - a}\cdot\left(1 + \frac{z-a}{\zeta-a} + \left(\frac{z-a}{\zeta-a}\right)^2 + \dots\right) \\ &= \frac{1}{\zeta - a} + \frac{z-a}{(\zeta-a)^2} + \frac{(z-a)^2}{(\zeta-a)^3} + \dots + \frac{(z-a)^n}{(\zeta-a)^{n+1}} + \dots\end{aligned}$$

Here, notice that $\frac{|z-a|}{|\zeta-a|} < 1$ since $\zeta \in \gamma_p$, which allows us to expand using the geometric series. Also, convergence is uniform on γ_p. By substitution, we have

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{\zeta - a} + \frac{f(\zeta)}{(\zeta-a)^2}(z-a) + \dots + \frac{f(\zeta)}{(\zeta-a)^{n+1}(z-a)^n} + \dots$$

where convergence is uniform for $\zeta \in \gamma_p$. Finally, we see that

$$\begin{aligned}f(z) &= \frac{1}{2\pi i}\int_{\gamma_p}\frac{f(\zeta)\,d\zeta}{\zeta - z} \\ &= \underbrace{\frac{1}{2\pi i}\int_{\gamma_p}\frac{f(\zeta)\,d\zeta}{\zeta - a}}_{f(a)} + \underbrace{\frac{1}{2\pi i}\int_{\gamma_p}\frac{f(\zeta)\,d\zeta}{(\zeta-a)^2}}_{f'(a)}(z-a) + \dots + \underbrace{\frac{f(\zeta)\,d\zeta}{(\zeta-a)^{n+1}}}_{f^{(n)}(a)}(z-a)^n + \dots\end{aligned}$$

Thus, we have the representation of $f(z)$ as a Taylor series.

□

Definition. *Analytic*
Suppose that $U \subseteq \mathbb{C}$ is open, $f : U \to \mathbb{C}$. One says that f is **analytic** if, for any $a \in U$, there exists a $D_{a,r} \subseteq U$ and there exists coefficients $c_0, c_1, \dots$ such that

$$f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + \dots$$

for any $z \in D_{a,r}$.

We have proved above theorem that any holomorphic function is analytic. The converse is also true. To show the converse, we will prove a stronger statement.

Theorem. *Analytic functions are holomorphic*
Suppose that $U \subseteq \mathbb{C}$ is open and let $\{f_n\}$ be a sequence of holomorphic functions on U such that $\lim_{n\to\infty} f_n = f$ uniformly on compact subsets of U. Then

1. f is holomorphic
2. $\lim_{n\to\infty} f_n' = f'$ uniformly on compact subsets of U.

Notice that neither consequent is true for $\mathscr{C}^\infty$ functions of a real variable.

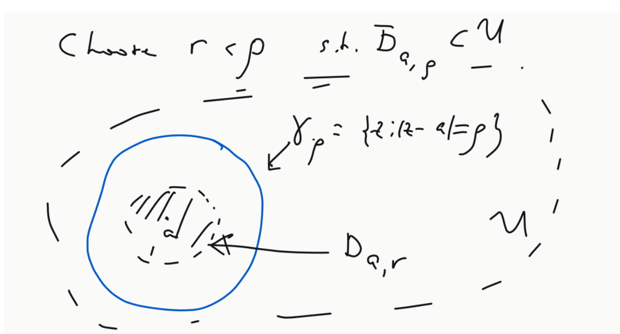
Aside. **Any continuous** function $[a,b] \to \mathbb{R}$ is the limit of a uniformly convergent sequence of polynomials. This is called the Stone-Weierstrass theorem. So much for (1) in the $\mathscr{C}^\infty$ context. To be explicit, the point here is that continuous real functions that are not so nice (i.e. Weierstrass' function) may be approximated by smooth functions on $\mathbb{R}$. Even if f is $\mathscr{C}^\infty$, it is not always possible to differentiate term-wise. Consider $f_n : \mathbb{R} \to \mathbb{R}$ given by

$$f_n(x) = \frac{\sin(n^2 x)}{n}$$

We have $f_n \to 0$ uniformly on $\mathbb{R}$, $|f_n(x)| \le \frac{1}{n}$ for any x. However, $f_n'(x) = n\cos(n^2 x)$ does not even converge pointwise!

Proof of the theorem.

1. We show $f_n \to f$ uniformly on compact subsets of U implies that f is holomorphic. Let $a \in U$. It suffices to check that f is holomorphic on some $D_{a,r} \subseteq U$. Choose $r < \rho$ such that $\overline{D}_{a,\rho} \subseteq U$.



Since f_n is holomorphic, for all $z \in D_{a,r}$ we may write

$$f_n(z) = \frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f_n(\zeta)\,d\zeta}{\zeta - z}$$

Also, we know that

- $f_n \to f$ uniformly on γ_ρ.
- For all $\zeta \in \gamma_\rho$, $z \in D_{a,r}$ we have $|\zeta - z| \geq \rho - r > 0$. This shows that

$$\frac{1}{|\zeta - z|} \leq \frac{1}{\rho - r}$$

Together, we have that

$$\frac{f_n(\zeta)}{\zeta - z} \xrightarrow[n\to\infty]{} \frac{f(\zeta)}{\zeta - z}$$

where convergence is uniform with respect to $\zeta \in \gamma_\rho$.

Aside. Uniform bounds. To show uniform convergence of $\frac{f_n(\zeta)}{\zeta-z} \to \frac{f(\zeta)}{\zeta-z}$, we use the bound on

$$\left|\frac{f_n(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - z}\right| = \left|\frac{f_n(\zeta) - f(\zeta)}{\zeta - z}\right|$$

- The numerator $|f_n(\zeta) - f(\zeta)| < \varepsilon$ uniformly on γ_ρ.
- The denominator $|\zeta - z|$ has a uniform lower bound $\rho - r$.
  So: $\left|\frac{f_n(\zeta)}{\zeta-z} - \frac{f(\zeta)}{\zeta-z}\right| \leq \frac{\varepsilon}{\rho-r}$

This inequality shows that the convergence of $\frac{f_n(\zeta)}{\zeta-z} \to \frac{f(\zeta)}{\zeta-z}$ is uniform. The uniform bound is important because it prevents the denominator from becoming too small.

Observe that

$$\frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f(\zeta)\,d\zeta}{\zeta - z} = \lim_{n\to\infty} \frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f_n(\zeta)\,d\zeta}{\zeta - z} = \lim_{n\to\infty} f_n(z) = f(z)$$

So, since we may represent f using a Cauchy-type integral, we conclude that f is holomorphic.

2. To begin, we know that $f_n \to f$ uniformly on compact subsets of U, and that f_n and f are holomorphic.

Step 1. We want to show that $f_n' \to f'$ uniformly on any $\overline{D}_{a,r} \subseteq U$. There exists a $\rho > r$ such that $\overline{D}_{a,r} \subseteq U$. For $z \in \overline{D}_{a,r}$, we have

$$f_n'(z) = \frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f_n(\zeta)\,d\zeta}{(\zeta - z)^2}, \quad f'(z) = \frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f(\zeta)\,d\zeta}{(\zeta - z)^2}$$

Again, we know

- $f_n \to f$ uniformly on γ_ρ.
- For all $\zeta \in \gamma_\rho$, $z \in \overline{D}_{a,r}$, we have

$$\left|\frac{1}{(\zeta - z)^2}\right| \leq \frac{1}{(\rho - r)^2}$$

Now, for $z \in \overline{D}_{a,r}$

$$|f_n'(z) - f'(z)| = \frac{1}{2\pi}\left|\int_{\gamma_\rho} \frac{f_n(\zeta) - f(\zeta)}{(\zeta - z)^2}\, d\zeta\right|$$
$$\leq \underbrace{\frac{1}{2\pi} \cdot 2\pi\rho \cdot \frac{1}{(\rho - r)^2}}_{\text{constant}} \cdot \sup_{\zeta \in \gamma_\rho} |f_n(\zeta) - f(\zeta)|$$

Notice that this supremum is independent of z and goes to zero.

Step 2. Let $K \subseteq U$ be compact. Any point of K is in the interior of a closed disk that is contained in U. Also, K is a subset of the union of a finite number of such disks. The finiteness is important because we can just take the maximum of all N_i across the disks to show uniform convergence on the entire set K.

Corollary. A function on an open subset of $\mathbb{C}$ is holomorphic if and only if it is analytic.

Proof. We know that a function is holomorphic if and only if it is analytic. We discuss the direction: analytic implies holomorphic. It suffices to assume that f may be represented as a power series on an open disk. If a power series converges pointwise on an open disk, then it converges uniformly on compact subsets of this this open disk. To be explicit, because this is a power series, it satisfies the following properties:

1. Pointwise Convergence on $D_{a,r}$. For every $z \in D_{a,r}$, the series converges to $f(z)$.

2. Uniform Convergence on Compact Subsets of $D_{a,r}$. For any compact subset $K \subset D_{a,r}$, the power series converges uniformly on K. This is a fundamental property of power series due to the Weierstrass M-test (the convergence is controlled uniformly by the radius of convergence r).

Our function on this open disk is the limit of partial sums of this series which converges uniformly on compact subsets of the disk. But these are partial sums of polynomials, and polynomials are holomorphic, so we have the uniform limit of a sequence of holomorphic functions.

□

Proposition. Suppose that f is holomorphic in a neighborhood of a point a and that $f(a) = 0$. Then **either**

1. $f \equiv 0$ in a possibly smaller neighborhood of a, **or**

2. There exists an $\varepsilon > 0$ such that $0 < |z - a| < \varepsilon \implies f(z) \neq 0$.

Proof. On some $D_{a,r}$

$$f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + \dots$$
$$f(a) = c_0 = 0$$

Case 1. All the coefficients $c_m = 0$. Then, $f \equiv 0$ on $D_{a,r}$.

Case 2. There exists at least one non-zero coefficient. Let $k = \min\{m \in \mathbb{N} : c_m \neq 0\}$. On $D_{a,r}$, we have

$$\begin{aligned} f(z) &= c_k(z-a)^k + c_{k+1}(z-a)^{k+1} + \dots \\ &= (z-a)^k \underbrace{\left(c_k + c_{k+1}(z-a) + c_{k+2}(z-a)^2 + \dots\right)}_{\text{converges on } D_{a,r}} \\ &= (z-a)^k g(z) \end{aligned}$$

Note that $g(z)$ is holomorphic on $D_{a,r}$ and $g(a) = c_k \neq 0$. So, there exists (by continuity) an $\varepsilon > 0$ such that

$$|z-a| < \varepsilon \implies g(z) \neq 0$$

Thus, $0 < |z-a| < \varepsilon \implies f(z) \neq 0$.

□

Remark. This also does not apply to smooth, real functions. Actually, for any closed $F \subseteq \mathbb{R}$, there exists an $f \in \mathscr{C}^\infty$ with $f : \mathbb{R} \to \mathbb{R}$ such that $F = f^{-1}(0)$. As an example, consider the set of points $F = \{\frac{1}{n} : n \in \mathbb{N}\}$. A smooth function whose zeroes are equal to this set is: Find such an example later.

7.3 Order

Definition. *Order*
Suppose f is holomorphic on a neighborhood of a and that $f \not\equiv 0$ on any neighborhood of a. Then

$$\operatorname{ord}_a f := \min\{k \in \mathbb{Z}_{\geq 0} : f^{(k)}(a) \neq 0\}$$

Examples.

1. $$f(z) = \frac{1}{1-z^3}, \quad \operatorname{ord}_0(f) = 0$$
2. $f(z) = z^2$, $\operatorname{ord}_0(f) = 2$.
3. $f(z) = z - \sin(z)$, $\operatorname{ord}_0(f) = 3$.
4. $f(z) = \sin(1 - \cos(z))$, $\operatorname{ord}_0(f) = 2$. To see this, write out the Taylor expansions fully and simplify to get the first few terms.
5. $$f(z) = \frac{1}{1 + \sin(z)}$$
   Find $\operatorname{ord}_a(f)$.

For the final example, we can observe that, given $\operatorname{ord}_a f = k$ and $\operatorname{ord}_a(g) = l$, we have $\operatorname{ord}_a(f \cdot g) = k + l = \operatorname{ord}_a(f) + \operatorname{ord}_a(g)$. This is since

$$\begin{aligned} f(z) &= (z-a)^k(c_k + c_{k+1}(z-a) + \ldots) \\ g(z) &= (z-a)^l(d_l + d_{l+1}(z-a) + \ldots \\ (f \cdot g)(z) &= (z-a)^{k+1}(c_k d_l + (c_k d_{l+1} + c_{k+1}d_l)(z-a) + \ldots) \end{aligned}$$

We also have

$$\operatorname{ord}_a(f+g) \begin{cases} \geq k, \text{ if } k = l, \\ = \min(k, l) \text{ if } k \neq l \end{cases}$$

Based on this, we can determine $\operatorname{ord}_a(f)$. A more difficult way to derive the same fact is to expand f as a geometric series:

$$\begin{aligned} \frac{1}{1-(-\sin(z))} &= 1 - \sin(z) + \sin^2(z) - \ldots \\ &= 1 - \left(z - \frac{z^3}{3!} + \ldots\right) + \left(z - \frac{z^3}{3!} + \ldots\right)^2 - \left(z - \frac{z^3}{3!} + \ldots\right)^3 + \ldots. \\ &= 1 - \left(z - \frac{z^3}{3!} + \ldots\right) + (z^2 - \ldots) \pm \ldots \\ &= 1 - z + z^2 - \left(\frac{1}{3!} + 1\right) z^3 \pm \ldots \end{aligned}$$

Plugging in $z = a$ will show us if $f(a) = 0$.

Aside. What do we currently know about holomorphic functions?

1. Equivalence with Cauchy-Riemann equations.
2. Holomorphic functions have the property of preserving angles.
3. Cauchy's integral theorem
$$\int_\gamma f(z)\, dz = 0$$
4. A holomorphic function in an open disc has a primitive in that disc.
5. Cauchy's integral formula
$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\, d\zeta}{\zeta - a}$$
The idea is values on the boundary determine values in the interior.
6. Conversely, a function that satisfies Cauchy's integral formula is holomorphic.
7. Regularity. If f is holomorphic in an open set U, then f has infinitely many complex derivatives in U. We know how to compute the values of those derivatives as well.
8. Morera's Theorem. Some conditions under which a function is holomorphic.
9. A function is holomorphic if and only if it is analytic.

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Recall: If f is holomorphic on $U \subseteq \mathbb{C}$, $f(a) = 0$, then either a is an isolated point of $\{z \in U : f(z) = 0\}$ or f is identically 0 in a neighborhood of a.
In the first case, one says that a is an isolated zero of f.

8.1 Identity Theorem and its Applications

Definition. *Disconnected, Connected*
An open subset $U \subseteq \mathbb{C}$ is called disconnected if $U = U_1 \cup U_2$, U_j are open $(j = 1, 2)$, $U_1 \cap U_2 = \varnothing$, and U_1 and U_2 are nonempty.
An open set $U \subseteq \mathbb{C}$ is called connected if it is **not disconnected.**

Fact. A open subset $U \subseteq \mathbb{C}$ is connected if and only if any two points of U can be joined by a path lying in U.

Proposition. Suppose $U \subseteq \mathbb{C}$ is open and connected and $f : U \to \mathbb{C}$ is holomorphic, and let

$$S = \{z \in U : f(z) = 0\}$$

If S has a cluster point in U, then $f \equiv 0$.

Proof. Suppose, for contradiction, $a \in U$ is a cluster point of S and $f \not\equiv 0$. Define

$$U_1 := \{b \in U : f(b) = 0, f^{(n)}(b) = 0\,, \forall n \in \mathbb{N}\}$$
$$U_2 := U \setminus U_1$$

By construction, $U_1 \cap U_2 = \varnothing$ and their union is U. We see that $U_2 \neq \varnothing$ since $f \not\equiv 0$. Further, U_2 is open since $f, f', f'', \ldots$ are continuous. To be explicit, suppose that $f(b) \neq 0$. Since f is continuous, there is some open set around b which f maps only to nonzero values. So, this open set is contained in U_2, showing that b is an interior point. Hence U_2 is open.
Now we consider U_1. Since a is a cluster point of S and $f(a) = 0$, this shows that a is not an isolated zero of f. So, there exists an open set $V \subseteq U$ containing a and which satisfies $f\big|_V \equiv 0$. Hence

$$f(b) = f^{(n)}(b) = 0, \ \forall b \in V, \ \forall n \in \mathbb{N}$$

So $a \in V \subseteq U_1$, showing that U_1 is nonempty. It remains to show that U_1 is open. Let $b \in U_1$. Since U is open, there exists some $D = \{z \in \mathbb{C} : |z - b| < r\}$ satisfying $b \in D \subseteq U$. Since f is holomorphic, then on D, we may write

$$f(z) = f(b) + f'(b)(z - b) + \frac{f''(b)}{2!}(z - b)^2 + \ldots$$

So, $f\big|_D \equiv 0$ and $D \subseteq U_1$, showing that b is an interior point.

□

Corollary 1. *Identity Theorem.* Suppose $U \subseteq \mathbb{C}$ is open and connected. $f, g : U \to \mathbb{C}$ are holomorphic, $S \subseteq U$ is a set having a cluster point in U, and $f\big|_S \equiv g\big|_S$. Then $f \equiv g$.

This corollary says that the values on a countable set of points is enough to determine the behavior on the domain.

Corollary 2. Suppose $U \subseteq \mathbb{C}$ is open and connected, $V \subseteq U$ is open and nonempty, $f, g : U \to \mathbb{C}$ is holomorphic, and $f\big|_V \equiv g\big|_V$. Then $f \equiv g$.

Corollary 2 is a more general form of the identity theorem that extends to holomorphic functions in $\mathbb{C}^n$, for $n > 1$. Corollary 1 is false in this setting. As an example, consider the function $f : \mathbb{C}^2 \to \mathbb{C}$ given by $f(z, w) = z - w$. Then $f\big|_\Delta \equiv 0$, where $\Delta = \{(t, t) \in \mathbb{C}^2\}$, but $f \not\equiv 0$.

8.1.1 Exercises

1. Let $D = \{z \in \mathbb{C} : |z| < 1\}$. Suppose $f : D \to \mathbb{C}$ is holomorphic and

$$f\left(\frac{1}{n}\right) = \frac{n+2}{n+3}$$

for any $n \in \mathbb{N}$, $n \geq 2$. What can be said about $f(\frac{2}{3})$?

Solution. As an easier problem, we can consider finding the value of $f(0)$. We know that $f(0) = 1$ by continuity. Now, let $S = \{\frac{1}{n} : n \in \mathbb{N},\ n \geq 2\}$. Simply consider the function $g : D \to \mathbb{C}$ given by

$$g(z) = \frac{\frac{1}{z} + 2}{\frac{1}{z} + 3}$$

Since $g\big|_S \equiv f\big|_S$, we may conclude that

$$f\left(\frac{2}{3}\right) = g\left(\frac{2}{3}\right) = \frac{\frac{3}{2} + 2}{\frac{3}{2} + 3} = \frac{7}{9}$$

2. Does there exist a holomorphic function $f : D \to \mathbb{C}$ such that $f(\frac{1}{n}) = \frac{(-1)^n}{n^2}$ for all $n \in \mathbb{N}$, $n > 1$?

Solution. Assume, for contradiction, that such a function f satisfying the conditions above. Put

$$S_1 = \left\{\frac{1}{2n+1} : n \in \mathbb{N}\right\}, \quad S_2 = \left\{\frac{1}{2n} : n \in \mathbb{N}\right\}$$

and define the functions $g_1(z) = -z^2$ and $g_2(z) = z^2$. Note that S_1 and S_2 have a cluster point $0 \in D$, so the assumptions of the identity theorem are established. But now $g_1\big|_{S_1} \equiv f\big|_{S_1}$ and $g_2\big|_{S_2} \equiv f\big|_{S_2}$ and $g_2 \not\equiv g_1$ ($\lightning$).

$\square$

8.2 Laurent Series

Suppose $0 \le r < R \le +\infty$ for $a \in \mathbb{C}$. An annulus is given by

$$U = \{z \in \mathbb{C} : r < |z - a| < R\}$$

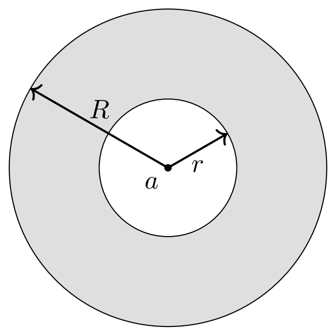

Figure 14: An annulus with inner radius r and outer radius R, centered at a.

Proposition. If, in the above setting, $f : U \to \mathbb{C}$ is holomorphic, then there exists an expansion

$$f(z) = \sum_{n\in\mathbb{Z}} c_n (z - a)^n \tag{$*$}$$

for all $z \in U$.

1. The series $(*)$ converges absolutely and uniformly on any compact $K \subseteq U$.
2. The expansion $(*)$ is unique.
3. We have
$$c_n = \frac{1}{2\pi i} \int_{\substack{|\zeta-a|=\rho,\\ r<\rho<R}} f(\zeta)(\zeta - a)^{-n-1}\, d\zeta$$

Proof.

1. *Uniqueness of the coefficients c_n.*
   Let $r < \rho < R$. Suppose that we have the representation
$$f(z) = \sum_{n\in\mathbb{Z}} c_n (z - a)^n$$

Multiply both sides by $(z-a)^{-k-1}$, for $k \in \mathbb{Z}$, we obtain

$$f(z)(z-a)^{-k-1} = \sum_{n\in\mathbb{Z}} c_n (z-a)^{n-k-1}$$

Note that the right-hand side converges uniformly on $\gamma_\rho = \{z \in \mathbb{C} : |z-a| = \rho\}$. Integrating term-wise, we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{\gamma_\rho} f(z)(z-a)^{-k-1}\,dz &= \frac{1}{2\pi i}\sum_{n\in\mathbb{Z}} c_n \int_{\gamma_\rho}(z-a)^{n-k-1}\,dz \\ &= \frac{1}{2\pi i} c_k \cdot 2\pi i = c_k\end{aligned}$$

Recall from a previous example that only $\int_C \frac{dz}{z} = 2\pi i$ remains, while the integral of other powers vanish.

2. *Existence of the Laurent expansion.*
   Choose ρ_1, ρ_2 so that $r < \rho_1 < |z-a| < \rho_2 < R$. Let $\gamma_1 = \{z \in \mathbb{C} : |z-a| = \rho_1\}$ and $\gamma_2 = \{z \in \mathbb{C} : |z-a| = \rho_2\}$. Let S be the area bounded by the curves γ_1 and γ_2.

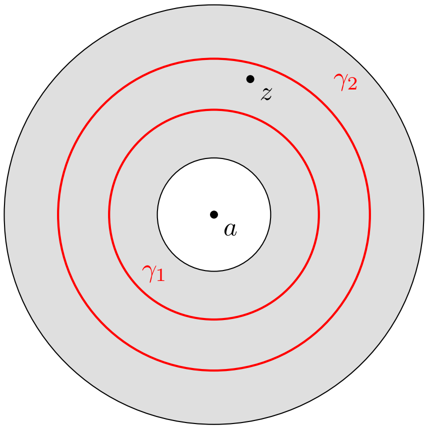

Figure 15: Annulus with γ_1 and γ_2

By Cauchy's integral formula, we have

$$\begin{aligned}f(z) &= \frac{1}{2\pi i}\int_S \frac{f(\zeta)\,d\zeta}{\zeta - z} \\ &= \underbrace{\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)\,d\zeta}{\zeta - z}}_{I_2} - \underbrace{\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)\,d\zeta}{\zeta - z}}_{I_1}\end{aligned}$$

Expand $\frac{1}{\zeta - z}$ on γ_2 and on γ_1 using geometric series. For I_2, we have

$$\begin{aligned}\frac{1}{\zeta - z} &= \frac{1}{(\zeta - a) - (z - a)} \\ &= \frac{1}{\zeta - a} \cdot \frac{1}{1 - \underbrace{\frac{z-a}{\zeta - a}}_{|\cdot|<1}} \\ &= \frac{1}{\zeta - a} \cdot \left(1 + \left(\frac{z-a}{\zeta-a}\right) + \left(\frac{z-a}{\zeta-a}\right)^2 + \dots\right)\end{aligned}$$

For I_1, we have a similar expression, except we need to consider that $|z-a| > |\zeta - a|$ since we are on the inner curve, γ_1.

$$\begin{aligned}\frac{1}{\zeta - z} = \frac{1}{(\zeta - a) - (z - a)} &= \frac{1}{z-a} \cdot \frac{1}{\underbrace{\frac{\zeta - a}{z - a}}_{|\cdot|<1} - 1} \\ &= -\frac{1}{z-a} \cdot \frac{1}{1 - \frac{\zeta - a}{z-a}} \\ &= -((z-a)^{-1} + (\zeta - a)(z-a)^{-2} + \dots)\end{aligned}$$

Now, observe that

$$-\frac{1}{2\pi i} \cdot I_1 = \left(\frac{1}{2\pi i} \cdot \int_{\gamma_1} f(\zeta)\, d\zeta\right)(z-a)^{-1} + \left(\frac{1}{2\pi i} \cdot \int_{\gamma_1} f(\zeta)(\zeta - a)\, d\zeta\right)(z-a)^{-2} + \dots$$

So adding both components together, we see that $f(z)$ has a Laurent expansion. Although the computation of the coefficients on *different* curves γ_1 and γ_2 is still consistent with the formula which involves only one radius because the Cauchy integral formula is independent of the closed curve used.

□

Definition. *Entire*

A function holomorphic on all of $\mathbb{C}$ is said to be *entire*.

9 November 15, 2024

9.1 Classification on Isolated Singularities

Let $a \in \mathbb{C}$ and $r > 0$. Denote the punctured disk centered at a with radius r by

$$D^*_{a,r} = \{z \in \mathbb{C} : 0 < |z - a| < r\}$$

Definition. *Isolated Singularity*
If $f : D^*_{a,r} \to \mathbb{C}$ is holomorphic, one says that f has an *isolated singularity* at a.

We may split the Laurent series for f into two parts

$$f(z) = \underbrace{\ldots + c_{-2}(z-a)^{-2} + c_{-1}(z-a)^{-1}}_{\text{principal part}} + \underbrace{c_0 + c_1(z-a) + c_2(z-a)^2 + \ldots}_{\text{regular part}}$$

Definition. *Types of Isolated Singularities*
The principal part plays a key role in determining the type of isolated singularity.

1. If the principal part is zero, one says that the singularity is removable.
2. If the principal part is nonzero and finite, one says that f has a pole at a.
3. If the principal part is infinite, one says that f has an essential singularity.

Theorem. *Riemann's Removable Singularity Theorem*
The following assertions are equivalent.

1. f has a removable singularity.
2. f extends to $D_{a,r} = \{z \in \mathbb{C} : |z - a| < R\}$ as a holomorphic function.
3. There exist $\varepsilon > 0$, $C > 0$ such that $|f(z)| \leq C$ whenever $0 < |z - a| < \varepsilon$.

Proof. We first show that ① $\iff$ ②. For the forward direction, suppose that f has a removable singularity. As a result, we may write

$$f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + \ldots$$

Then, by setting $f(a) = c_0$, we see that f is analytic on $D_{a,r}$. Hence, it is holomorphic on $D_{a,r}$. In the other direction, if f is holomorphic on $D_{a,r}$, then it is analytic, so it has a removable singularity.
We now show ② $\implies$ ③. The extended function $f : D_{a,r} \to \mathbb{C}$ is holomorphic, this means that it is continuous at a. Hence, f must be bounded in a neighborhood of a (by continuity,

we may select a sufficiently small $\varepsilon > 0$ around a to bound values in the image set).

Finally, we show ③ $\implies$ ①. Let

$$c_{-n} = \frac{1}{2\pi i} \underbrace{\int_{|z-a|=\rho} f(z)(z-a)^{n-1}\, dz}_{I}$$

where $n > 0$ and $0 < \rho < \varepsilon$. Using the trivial estimate, we see that

$$|I| \leq 2\pi\rho \cdot C \cdot \rho^{n-1} = 2\pi C \rho^n \xrightarrow[\rho \to 0]{} 0$$

Here, we may consider the limit as $\rho \to 0$ because the value of I should be independent of the closed curve we use, given that we consider homotopic curves. This shows that the principal part vanishes, so f has a removable singularity.

□

9.2 Characterization of Poles

Definition. *Order of a Pole*
Suppose that f has a pole at a and

$$f(z) = \sum_{k \geq -n} c_k (z-a)^k, \quad c_{-n} \neq 0,\, n > 0$$

Then, one says that f has a pole of order $-n$:

$$\operatorname{ord}_a(f) = -n$$

or, informally, that f has a pole of order n.

Equivalently to the above definition, we may say that $f : D^*_{a,r} \to \mathbb{C}$ has the form $f(z) = \frac{g(z)}{(z-a)^n}$ for a positive integer n and a function g holomorphic in the neighborhood of a with $g(a) \neq 0$.

Proposition. The following assertions are equivalent.

1. f has, at a, a removable singularity or a pole of order $\geq -n$, for $n > 0$.
2. $|f(z)| = O(|z-a|^{-n})$ as $z \to a$. To be explicit, there exists $\varepsilon > 0$, $C > 0$ such that $|f(z)| \leq C \cdot |z-a|^{-n}$ whenever $0 < |z-a| < \varepsilon$.

Proof. We first show ① $\implies$ ②. Letting $k \leq n$, without loss of generality, we have

$$\begin{aligned} f(z) &= c_{-k}(z-a)^{-k} + c_{-k+1}(z-a)^{-k+1} + \ldots \\ &= (z-a)^{-k} \underbrace{\left(c_{-k} + c_{-k+1}(z-a) + c_{-k+2}(z-a)^2 + \ldots\right)}_{g(z),\ \text{holomorphic on } D_{a,r}} = \frac{g(z)}{(z-a)^k} \end{aligned}$$

Since $g(z)$ is holomorphic at a, it is bounded near a. Hence $|f(z)| = O(|z-a|^{-k}) = O(|z-a|^{-n})$.
To show that ② $\implies$ ①, assume that $|f(z)| = O(|z-a|^{-n})$. Then, there exist $C > 0$ and $\varepsilon > 0$ such that

$$0 < |z-a| < \varepsilon \implies |f(z)| \leq \frac{C}{|z-a|^n}$$

Put $g(z) := (z-a)^n f(z)$, then we see that $0 < |z-a| < \varepsilon \implies |g(z)| < C$. Now, by Riemann's theorem on removable singularities, we conclude that $g(z)$ has a removable singularity. Observe that

$$g(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + \dots$$
$$f(z) = c_0(z-a)^{-n} + c_1(z-a)^{-n+1} + \dots$$

Hence, f has a pole of order at most n.

□

In the above proof, note that it is possible for f to have a removable singularity when g has a removable singularity if $f \equiv 0$.
Remark. If f and g have both at worst a pole at a, then this is the case for $f+g, f-g, f \cdot g, f/g$ (i.e. these have at worst a pole). The claim is clear for the first two expressions by inspecting the sum of the Laurent series. For the latter expressions, consider

$$\sum_{j \geq -m} c_j z^j \cdot \sum_{k \geq -n} b_k z^k$$

For division, suppose $m \in \operatorname{ord}_a(f)$ and $n \in \operatorname{ord}_a(g)$. Let $f(z) = (z-a)^m f_1(z)$ with $f_1(a) \neq 0$ and $g(z) = (z-a)^n g_1(z)$ with $g_1(a) \neq 0$. We may write

$$\frac{f(z)}{g(z)} = (z-a)^{m-n} \cdot \frac{f_1(z)}{g_1(z)}$$

where $\frac{f_1(z)}{g_1(z)}$ is holomorphic in a neighborhood of a.
Remark. If both f and g have at worst a pole at a, then

$$\operatorname{ord}_a(fg) = \operatorname{ord}_a(g) + \operatorname{ord}_a(f), \quad \operatorname{ord}_g\left(\frac{f}{g}\right) = \operatorname{ord}_a(f) - \operatorname{ord}_a(g)$$

9.3 Meromorphic Functions and Some Theorems

Definition. *Meromorphic*
Suppose $U \subseteq \mathbb{C}$ is open. A *meromorphic* function f is a holomorphic function $f : U \setminus S \to \mathbb{C}$ (where $S \subseteq U$ is discrete) having at worst poles at points of S.

Theorem. *Casorati-Weierstrass Theorem*
The following assertions are equivalent.

1. a is an essential singularity.

2. For any $c \in \mathbb{C} \cup \{\infty\}$ there exists a sequence $a_n \to a$ such that $f(a_n) \to c$.

Proof. We first show ② $\implies$ ①. Suppose that a is not an essential singularity. Since a is not an essential singularity, it must be at worst a pole.

- Suppose a is removable. Then we have $\lim_{z\to a} f(z) = f(a)$.

- Suppose a is a pole. Then, there exists $f_1(z)$ such that $f_1(a) \neq 0$ and $f(z) = \frac{f_1(z)}{(z-a)^n}$. Thus

$$\lim_{z\to a} f(z) = \infty$$

As all sequences $a_n \to a$ result in $f(a_n)$ converging to only one value in both cases, we conclude ② is false.
Now, suppose that there exists some $c \in \mathbb{C} \cup \{\infty\}$ such that for all sequences $z_n \to a$, we have $\lim_{n\to\infty} f(z_n) \neq c$.

- Suppose $c \in \mathbb{C}$. Since $\lim_{n\to\infty} f(z_n) \neq c$, there exists $\delta > 0$ such that for all $\varepsilon > 0$ we have

$$0 < |z - a| < \varepsilon \implies |f(z) - c| \geq \delta$$

  Put $g(z) = \frac{1}{f(z)-c}$. It follows that

$$0 < |z - a| < \varepsilon \implies |g(z)| \leq \frac{1}{\delta}$$

  So, g has a removable singularity by Riemann's removable singularity theorem. Rearranging, we see that $f(z) = c + \frac{1}{g(z)}$. Since f may be obtained from arithmetic operations of functions that have at worst a pole at a, this shows that f has at worst a pole at a.

- Suppose $c = \infty$. Then $\lim_{n\to\infty} f(z_n) \neq \infty$ means that $f(z)$ is bounded in some neighborhood around a. So, f has a removable discontinuity at a.

□

Theorem. *Liouville's Theorem*
Suppose f is a holomorphic function on $\mathbb{C}$ and $|f(z)| = O(|z|^N)$ as $|z| \to \infty$. Then, f is a polynomial of degree at most N.

Aside. The notation $|f(z)| = O(|z|^N)$ means that there exists $M > 0$, $C > 0$ such that

$$|z| \geq M \implies |f(z)| \leq C \cdot |z|^N$$

According to online sources, this is also equivalent to

$$\limsup_{z \to a} \frac{|f(z)|}{|z^N|} < \infty$$

Proof. Put $g(z) = f(\frac{1}{z})$. We see that g is holomorphic on $\mathbb{C}^*$ and that it has an isolated singularity at 0. Define $\varepsilon = \frac{1}{M}$, and assume $0 < |z| \leq \varepsilon$. Rearranging, we see that $|\frac{1}{z}| \geq M$, and by our assumption that $|f(z)| = O(|z|^N)$, it follows that there exists a $C > 0$ such that

$$\left| f\left(\frac{1}{z}\right) \right| \leq \frac{C}{|z|^N}$$

By substitution, we have $|g(z)| \leq \frac{C}{|z|^N}$. So, $|g(z)| = O(|z|^{-N})$ as $z \to 0$, showing that $g(z)$ has at worst a pole at 0, or $\text{ord}_0(g) \geq -N$. Now

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots$$
$$g(z) = f\left(\frac{1}{z}\right) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots$$

Since we know that g has at worst a pole of order $-N$, we have $a_j = 0$ for $j > N$. To be explicit, we also know that $a_{-k} = 0$ for $k \in \mathbb{N}$ since f is holomorphic, which is why these terms are not shown. The desired conclusion follows.

□

Corollary. A bounded entire function is constant.

Proof. Apply Liouville's theorem with $N = 0$.

□

Examples. Find and classify the singularities for

1. $$f(z) = \sin\left(\frac{1}{z}\right)$$

2. $$f(z) = \frac{z}{e^z - 1}$$

10 November 22, 2024

10.1 Residues

Suppose f is a holomorphic function with an isolated singularity at $a \in \mathbb{C}$.

Definition. *Residue*
Given the above, we may write

$$\operatorname{Res}_a f(z)\,dz = \frac{1}{2\pi i}\int_{|z-a|=\varepsilon} f(z)\,dz$$

Example. Suppose f has a removable singularity at a. We have

$$0 = c_{-1} = \frac{1}{2\pi i}\int_{|z-a|=\varepsilon} f(z)\,dz = \operatorname{Res}_a f(z)\,dz$$

Another way to see this is to use the fact that f may be extended to a holomorphic function on a disk around a, and then Cauchy's integral theorem yields this result.

Proposition. If

$$f(z) = \sum_{n\in\mathbb{Z}} c_n (z-a)^n$$

then $\operatorname{Res}_a f(z)\,dz = c_{-1}$.

Proof. This follows by direct comparison with the value of c_{-1} given in the proposition regarding the existence of the Laurent series on annuli.
An alternate way to arrive at this conclusion is to integrate over a closed curve γ where $\operatorname{ind}_a(\gamma) = 1$, in this case, we see that

$$\int_\gamma f(z)\,dz = \sum_{n\in\mathbb{Z}} c_n \int_\gamma (z-a)^n\,dz = 2\pi i \cdot c_{-1}$$

□

Theorem. *Residue Theorem.*
Suppose that $D \subseteq \mathbb{C}$ is open, bounded, and has a piecewise smooth boundary. Let $S \subseteq D$ be finite and f be a function that is continuous on $\overline{D} \setminus S$ and holomorphic on $D \setminus S$. Then

$$\int_{\partial D} f(z)\,dz = 2\pi i \sum_{a\in S} \operatorname{Res}_a f(z)\,dz$$

11 Appendix

We list and verify some statements made or assumed during lecture.

1. Context. Given a complex valued function φ, we have
$$\left|\int_p^q \varphi(t)\,dt\right| \leq \int_p^q |\varphi(t)|\,dt$$
I remember there is a proof using Cauchy-Schwarz for this? But when we talked in class we assumed it to be true?

2. A linear fractional transformation may be identified with a matrix up to scaling by a nonzero constant.
Recall that a linear fractional transformation is given by the mapping $z \mapsto \frac{az+b}{cz+d}$ for $a, b, c, d \in \mathbb{C}$ and where $ad - bc$ is nonzero. It turns out that the set of linear fractional transformations corresponds to the group $\mathrm{PSL}(2, \mathbb{C})$ (?)
$$\mathrm{SL}(2, \mathbb{C}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C}, \det(A) = ad - bc = 1 \right\}$$
$$\mathrm{PSL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C}) \big/ \{\pm I_2\}$$
Note that not all scalings are captured because we are considering matrices with determinant 1. Why do we fix the determinant?

3. Context. A composition of two linear fractional transformations is again a linear fractional transformation. The inverse to a linear fractional transformation is a linear fractional transformation.
The first statement is a straightforward verification we have already carried out in Homework 5. For the second statement, consider
$$x = \frac{ay+b}{cy+d} = A + \frac{B}{y+C}$$
$$y = \frac{B}{x-A} - C$$

4. A homeomorphism maps interiors to interiors and boundaries to boundaries.
Useful link. I suppose this is not exactly complex analysis and more about topology.

5. Context. Why does the convention for symmetry of points on generalized circles make sense?
The point of the convention is not immediately clear to me.

6. Context. Are the details I filled in for this proof correct? About this proof, I was wondering why we mapped a circle to a line, instead of a circle to a circle?

It seems like it is related to making it easier to determine a symmetric point that we claim exists. The choice of mapping does not really matter in that regard, it is for convenience?

7. Context. Given $f(z) = \frac{z-a}{1-\bar{a}z}$ with the appropriate conditions, it is enough to show that $f(\partial D) = \partial D$ to prove that $f(D) = D$.
   The idea here appears to be that D must be mapped to either D, ∂D, or $\mathbb{C} \setminus \overline{D}$? To be clear, I understand the point of the argument saying that there is no point in D that maps to ∞ and I understand the structure of the argument which follows a process of elimination - it is not in $\mathbb{C} \setminus \overline{D}$ and it is not in ∂D, so it must be in what we have left, D! But I remain skeptical of how we got to this setup in the first place. What property of the function is allowing us to reduce this problem to thinking about mapping into these three different sections?

8. Context. If f is a conformal mapping, then $f'(z) \neq 0$ for any $z \in U$ and $f^{-1} : V \to U$ is also holomorphic.

9. Context. In the example regarding the mapping $f(z) = z + \frac{1}{z}$, we have

$$\begin{aligned}\mathbb{C} \setminus f(U) &= \left\{a \in \mathbb{C} \mid \forall z \in U : z + \frac{1}{z} \neq a\right\} \\ &= \{a \in \mathbb{C} : |z_1| = 1 \text{ and } |z_2| = 1\}\end{aligned}$$

   Explanation. We determined that a solution to $z + \frac{1}{z} = a$ must occur in pairs z_1, z_2, as the roots of a quadratic equation. We want roots such that both z_1 and z_2 are not in U, and such roots will determine a. Vieta's formulas show us that $z_1 z_2 = 1$. Because $|z_1| < 1$ means $|z_2| > 1$, this implies that $a \in f(U)$ for both of these roots. In particular, there exists some $z \in U$ such that $f(z) = a$. Thus, the complement must consist of roots where $|z_1| = 1$ and $|z_2| = 1$.

10. Linear fractional mappings are conformal on $\overline{\mathbb{C}}$.

11. Context. In class, you began to prove Morera's Theorem using one method, but abandoned the explanation for a quicker and more direct explanation. I filled in the details of what I thought you were saying in an alternative approach and I wanted to check if it is correct.

12. In the proposition for Cauchy-Type integrals, we should check the claims of uniform convergence.

13. Cauchy's integral formula is independent of the closed curve that is used? Yes.

14. If we only know that a function is holomorphic at a point, do we know anything else about it?